Chapter 16

Open and closed sets

Introduction

In this chapter we introduce some ideas that help us to describe particular important kinds of subsets that occur in metric spaces. This provides us with some useful terminology that we will make extensive use of in Book F, where we return to our study of metric spaces. Metric spaces can be very abstract objects and having terms for particular types of subsets can help us to understand their structure better.

Our starting point for this is the main tool we have used for investigating and understanding metric spaces, namely convergent sequences. We start this chapter by looking at sets that behave 'nicely' with regard to convergent sequences. In particular, we look at sets that contain the limit points of those convergent sequences lying within them – such sets are known as *closed sets* and they turn out to play an important role in some of the theorems that we will prove in Book F. We investigate the properties of closed sets and look at how to determine them for particular metric spaces.

We also look at another important class of subsets – the *open sets*. These sets occur as the complements of closed sets, and as a result there are close connections between their properties and the properties of closed sets.

Using the notion of *closure*, which we introduce in Section 3, we find a way to associate a 'natural' closed set with any set. It is also possible to associate an open set with any given set and we investigate this in Section 4, where we introduce the notion of the *interior* of a set.

In Section 5 we discuss how to find the *boundary* of an arbitrary set, which consists of those points that are (arbitrarily) near to both the set and its complement.

Section 6 takes a slightly different perspective from the rest of the chapter and looks at two ways of measuring the 'size' of sets. We first look at a metrical notion of large sets – those whose closure equals the whole space. In the second half of the section, we look at a notion of size that is purely set-theoretic and does not use the metric (should one be available). This leads us to discovering that the set of real numbers is strictly 'bigger' than the set of natural numbers.

In the final section, which is not assessed, we discuss how the ideas of this chapter lead naturally to the notion of a *topology*.

Chapter guide

This chapter contains many concepts but, with the exception of the discussion of countable and uncountable, they all rely on the notion of a closed set, which is defined in the first section, and so this is the key idea to understand.

If you are short of time, then you should concentrate your efforts on the first four sections of the chapter; you may wish to omit the last section, which is not directly assessed.

1 Closed sets

In Definition 1.4 of Chapter 15 we defined the notion of the induced or subspace metric for a subset $A \subseteq X$. This is the metric for A given by $d_A(a, b) = d(a, b)$ for $a, b \in A$. From the definition of the subspace metric and the definition of convergence for a sequence in the original metric space (X, d), we can see what it means for a sequence in (A, d_A) to be d_A-convergent:

> A sequence (a_n) in A is d_A-convergent if there is $a \in A$ for which $(d(a_n, a))$ is a real null sequence.

So suppose that (X, d) is a metric space and that A is a subset of X. Let (a_n) be a convergent sequence with respect to the metric d, all of whose terms are in A. Is the limit a of the sequence necessarily in A?

Exercise 1.1

For each of the following metric spaces and sets A, state whether or not the limit of the given convergent sequence is in the set A.

(a) Suppose that the space is $\mathbb{R}$ with the Euclidean metric, the sequence is $(\frac{1}{n})$ and the set is

(i) $A = [0, 1]$

(ii) $A = (0, 1)$.

(b) Suppose that the space is $(\mathbb{R}^2, e_1)$, the plane with the taxicab metric, the sequence $(\mathbf{a}_n)$ is given by $\mathbf{a}_n = (1 + \frac{1}{2n}, 2 - \frac{1}{n})$ for $n \in \mathbb{N}$ and the set is

(i) $A = B_{e_1}[(0, 1), 2]$

(ii) $A = B_{e_1}((0, 1), 2)$.

By Exercise 3.1 of Chapter 14 we know that this sequence converges to $(1, 2)$.

Exercise 1.1 demonstrates that a sequence that is convergent in the original metric space need not be convergent (with respect to the subspace metric) in a subspace that contains all the terms of the sequence.

Subsets of a metric space that (in the subspace metric) do contain the limits of convergent sequences are known as closed sets.

Definition 1.1 *Closed set*

Let (X, d) be a metric space and let $A \subseteq X$.

We say that A is a **d-closed set** if for each d-convergent sequence with terms in A, the limit of the sequence is also in A. That is, whenever there is a d-convergent sequence (a_n) with $a_n \in A$ for each n then $\lim_{n\to\infty} a_n \in A$.

Remarks

1. When it is clear from the context, we may suppress reference to the metric d and just refer to A being **closed**.
2. The definition of closed can be described in terms of the subspace metric on A: the set A is a d-closed set if, and only if, for each sequence (a_n) in A, (a_n) is d_A-convergent if, and only if, (a_n) is d-convergent. (Notice that it is always the case that a d_A-convergent sequence in the subspace is d-convergent in the original space, and so d_A-convergence only has to be tested for those sequences in A that are convergent in the larger space.)

1.1 Examples of closed sets

We will shortly look at closed sets in some specific metric spaces, but first we identify some sets that are always closed.

Lemma 1.2

Let (X, d) be a metric space. The following subsets of X are always d-closed:

1. the empty set, $\emptyset$
2. the whole space, X
3. sets consisting of a single point $\{x\}$ for some $x \in X$.

Proof

1. For the empty set, there are no sequences whose terms lie in it and so there is no condition to test. Hence it is closed. (This is an example of a result that holds purely because there is nothing that can fail to satisfy the condition to be tested.)
2. We need to show that if (a_n) is a sequence in X that is d-convergent (to x, say), then the limit is in X – but this is exactly the definition of convergence in (X, d) and so X is d-closed.
3. Let $x \in X$ be given and let $A = \{x\}$. If (a_n) is a sequence in A, then $a_n = x$ for each $n \in \mathbb{N}$ and so in particular is an eventually constant sequence. Hence by Worked Exercise 3.5 of Chapter 14, (a_n) is d-convergent with limit x, which is in A. Thus A is d-closed as claimed. ■

Our next standard (and slightly more interesting) examples of closed sets are the closed balls in a metric space, which at least shows that our choice of terminology is consistent!

Recall that
$B_d[x, r] = \{y \in X : d(y, x) \leq r\}$.

Proposition 1.3

Let (X, d) be a metric space, let $x \in X$ and let $r \geq 0$. Then the closed ball $B_d[x, r]$ is a d-closed set.

Proof Let (a_n) be a convergent sequence in X with limit a, and suppose that $a_n \in B_d[x, r]$ for each $n \in \mathbb{N}$. In order to show that the ball $B_d[x, r]$ is d-closed we need to show that $a \in B_d[x, r]$. Thus our goal is to show that $d(a, x) \leq r$.

Now, for each $n \in \mathbb{N}$, the Triangle Inequality for d and the fact that $a_n \in B_d[x, r]$ gives

$$d(a, x) = d(x, a) \leq d(x, a_n) + d(a_n, a) \leq r + d(a_n, a).$$

But $(d(a_n, a))$ is a null sequence and so for each $\varepsilon > 0$, there is $n \in \mathbb{N}$ for which $d(a_n, a) < \varepsilon$. Hence

$$d(a, x) < r + \varepsilon, \text{ for each } \varepsilon > 0.$$

In other words, $d(a, x) \leq r$, and so $a \in B_d[x, r]$ and the closed ball $B_d[x, r]$ is d-closed. ■

In particular, closed intervals of the form $[a, b]$ for $a, b \in \mathbb{R}$ with $a \leq b$ are closed sets for the Euclidean metric because
$[a, b] = B_{d^{(1)}}[\frac{1}{2}(a + b), \frac{1}{2}(b - a)]$.

Worked Exercise 1.4

Let $a \in \mathbb{R}$ and let d denote the Euclidean metric for $\mathbb{R}$. Show that $(-\infty, a]$ is a d-closed set.

Solution

Let (x_n) be a d-convergent sequence in $\mathbb{R}$ with limit x, and suppose that $x_n \in (-\infty, a]$ for each $n \in \mathbb{N}$. In order to show that the interval $(-\infty, a]$ is d-closed we need to show that $x \in (-\infty, a]$. Thus our goal is to show that $x \leq a$.

So suppose instead that $x > a$; then $|x - x_n| = x - x_n \geq x - a > 0$ for each $n \in \mathbb{N}$. Thus if we choose $\varepsilon > 0$ for which $x - a > \varepsilon$, then we have $|x - x_n| > \varepsilon$ for each $n \in \mathbb{N}$, contradicting the fact that $x_n \to x$ as $n \to \infty$. Hence $x \leq a$ and $(-\infty, a]$ is d-closed as required.

Exercise 1.2

Let $a \in \mathbb{R}$ and let d denote the Euclidean metric for $\mathbb{R}$. Show that $(-\infty, a)$ is not a d-closed set.

Although the interval $(-\infty, a)$ is not closed for the Euclidean metric, its *complement* is. Before asking you to show this, we recall the definition of the complement of a set.

Definition 1.5 *Complement of a set*

Let A be a subset of a set X. The **complement of A with respect to X** is $A^c = X - A$, the set of elements that are in X but not in A (Figure 1.1).

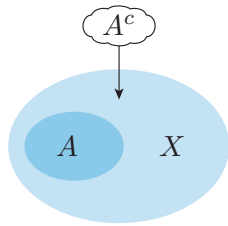


Figure 1.1 A^c is the complement of A in X

Remarks

1. The meaning of A^c depends on the context: we need to know the set X with respect to which we are taking the complement.
2. We can deduce immediately from the definition that $(A^c)^c = A$.

Exercise 1.3

Write down the complement A^c of $A = \{1, 3\}$ with respect to X when

(a) $X = \{1, 2, 3\}$; (b) $X = \{1, 2, 3, 4, 5\}$.

Exercise 1.4

Let $X = \mathbb{R}$ and $A = [0, 1]$. Write the complement A^c of A with respect to X as a union of intervals.

You are now asked to show that the complement of the interval $(-\infty, a)$ is closed for the Euclidean metric.

Exercise 1.5

Let $a \in \mathbb{R}$ and let d denote the Euclidean metric for $\mathbb{R}$. Show that the complement of $(-\infty, a)$ in $\mathbb{R}$ is a d-closed set.

Worked Exercise 1.6

Let (X, d) be a metric space. Show that for every choice of $x \in X$ and $r > 0$, the complement of the 'open' ball $B_d(x, r)$, $X - B_d(x, r)$, is d-closed.

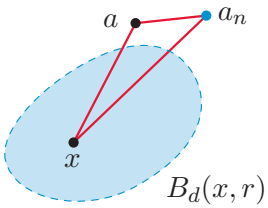


Figure 1.2 Showing that $X - B_d(x,r)$ is closed

Solution

Let $x \in X$ and $r > 0$ and suppose that the sequence (a_n) is such that $a_n \in X - B_d(x,r)$ for each $n \in \mathbb{N}$. That is, for each $n \in \mathbb{N}$, $d(a_n, x) \geq r$. If the sequence (a_n) is convergent for d with limit a, then the Reverse Triangle Inequality (Proposition 2.2 in Chapter 14) gives that for each $n \in \mathbb{N}$,

$$d(x,a) \geq |d(a_n,x) - d(a_n,a)| \geq r - d(a_n,a),$$

as illustrated in Figure 1.2. But $(d(a_n,a))$ is a real null sequence and so

$$d(x,a) \geq r,$$

which means that $a \notin B_d(x,r)$. That is, $a \in X - B_d(x,r)$, as required.

Exercise 1.6

Let (X,d) be a metric space, and $x \in X$ and $r \geq 0$ be given. Show that $A = \{y \in X : d(x,y) \geq r\}$ is a d-closed set.

We have now seen that intervals of the form $[a,b]$, $[a,\infty)$ or $(-\infty,a]$, which we already termed as being 'closed', are indeed closed for the Euclidean metric. Moreover, Exercise 1.5 shows that the complement of what we would usually term an 'open' interval, $(-\infty,a)$, is in fact d-closed: this turns out to be important when we come to define the notion of open sets (for a metric d) later in this chapter.

Worked Exercise 1.7

Show that $\mathbb{Q}$ is not a closed subset of $\mathbb{R}$ for the Euclidean metric.

Solution

To do this it is enough to find a sequence of rational numbers that converges to a point that is not rational. An example of such a sequence is given by (x_n), where x_n is the rational number obtained by taking the first n digits in the usual decimal representation of $\sqrt{2}$.

So $x_1 = 1$, $x_2 = 1.4$, $x_3 = 1.41$, ...

We now look at some other examples of closed sets for some of the metric spaces that we have met in previous chapters.

Closed sets in the plane

We have already seen that, for $\mathbb{R}$ with the usual Euclidean metric, sets of the form $[a, b]$, $(-\infty, a]$ and $[a, \infty)$ (for $a, b \in \mathbb{R}$) are closed. We also know that $\mathbb{R}$ itself is closed, as are sets consisting of a single point of $\mathbb{R}$. We have also seen some examples of sets that are not closed, such as the (infinite) 'open' interval $(-\infty, a)$ (for $a \in \mathbb{R}$), the 'open' interval $(0, 1)$ (see Exercise 1.1) and $\mathbb{Q}$.

We now look at examples of closed sets in the plane $\mathbb{R}^2$ (with the usual notion of distance, $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$). We already know from Proposition 1.3 that closed balls $B_d[\mathbf{x}, r]$ and the whole of the plane are closed sets for the Euclidean metric, so we are looking for some other examples of closed sets.

Before looking at our first examples, we record a useful lemma concerning convergent real sequences.

Lemma 1.8

Let (a_n) be a convergent sequence of real numbers with limit a (for the usual Euclidean metric). If $m, M \in \mathbb{R}$ are such that for each $n \in \mathbb{N}$, $m \leq a_n \leq M$, then

$$m \leq a \leq M.$$

Exercise 1.7

Prove Lemma 1.8.

Worked Exercise 1.9

Consider the plane $\mathbb{R}^2$ with the usual Euclidean metric d.

(a) Show that

$$\begin{aligned} U &= (0, 1) \times (0, 3) \\ &= \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1 \text{ and } 0 < x_2 < 3\} \end{aligned}$$

is not a d-closed set.

(b) Show that

$$\begin{aligned} H &= [0, 1] \times [0, 3] \\ &= \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq 3\} \end{aligned}$$

is a d-closed set.

U and H are illustrated in Figure 1.3.

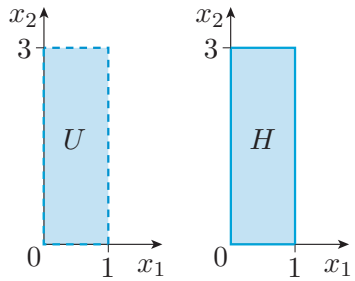


Figure 1.3 The sets U and H

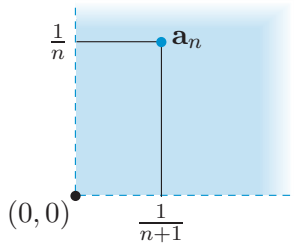


Figure 1.4 $(\mathbf{a}_n) = (\frac{1}{n+1}, \frac{1}{n})$

Solution

(a) We need only find a sequence whose terms lie in U that converges to a point outside U. The points on the 'boundary' of U are not in U and so we try a sequence that converges to a point on the boundary – define the sequence $(\mathbf{a}_n)$ by

$$a_n = (\tfrac{1}{n+1}, \tfrac{1}{n}) \text{ for } n \in \mathbb{N},$$

as shown in Figure 1.4. Clearly, since $0 < \frac{1}{n+1} < 1$ and $0 < \frac{1}{n} \leq 1 < 3$ for each $n \in \mathbb{N}$, $\mathbf{a}_n \in U$ for each $n \in \mathbb{N}$. However, $\mathbf{a}_n \to (0,0) \notin U$ as $n \to \infty$ for the usual Euclidean distance. Hence U is not closed for the Euclidean metric.

(b) In this case, we need to show that whenever $(\mathbf{a}_n)$ is a convergent sequence whose terms all lie in H, then the limit of the sequence (with respect to the Euclidean metric) also lies in H.

So suppose $(\mathbf{a}_n) = ((a_{1,n}, a_{2,n}))$ is such a sequence with limit $\mathbf{a} = (a_1, a_2)$ – we must show that $\mathbf{a} \in H$.

For each $n \in \mathbb{N}$,

$$0 \leq a_{1,n} \leq 1 \text{ and } 0 \leq a_{2,n} \leq 3.$$

Moreover, since $(\mathbf{a}_n)$ is convergent, we know from Theorem 3.15 of Chapter 15 that so are both of the real sequences $(a_{1,n})$ and $(a_{2,n})$ with limits a_1 and a_2, say. We can now immediately use Lemma 1.8 to deduce that

$$0 \leq a_1 \leq 1 \text{ and } 0 \leq a_2 \leq 3,$$

implying that $\mathbf{a} = (a_1, a_2) \in H$ as required.

The results of Worked Exercise 1.9 can be generalised to show that any set of the form $(a,b) \times (c,d)$, often referred to as an *open rectangle*, is not closed for the Euclidean metric on the plane and that any set of the form $[a,b] \times [c,d]$, often referred to as a *closed rectangle*, is closed.

Worked Exercise 1.10

Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $r > 0$. Show that for the plane with the usual Euclidean metric d, the open ball $B_d(\mathbf{x}, r)$ is not closed.

Solution

Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $r > 0$, and consider the sequence $(\mathbf{a}_n)$ given by $\mathbf{a}_n = \left(x_1 + r\left(1 - \frac{1}{n+1}\right), x_2\right)$ for $n \in \mathbb{N}$.

Then for each $n \in \mathbb{N}$,

$$d(\mathbf{a}_n, \mathbf{x}) = \sqrt{r^2(1 - \tfrac{1}{n+1})^2 + 0^2} = r(1 - \tfrac{1}{n+1}) < r$$

and so $\mathbf{a}_n \in B(\mathbf{x}, r)$ for each $n \in \mathbb{N}$.

However,

$$d(\mathbf{a}_n, (x_1 + r, x_2)) \to 0 \text{ as } n \to \infty,$$

meaning that the sequence $(\mathbf{a}_n)$ converges to $(x_1 + r, x_2)$ as $\mathbf{n} \to \infty$ (Figure 1.5). But $d(\mathbf{x}, (x_1 + r, x_2)) = r$ and so $(x_1 + r, x_2) \notin B_d(\mathbf{x}, r)$. Hence $B_d(\mathbf{x}, r)$ is not closed.

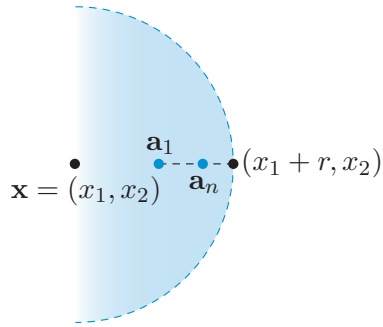


Figure 1.5 Sequence converging to $(x_1 + r, x_2)$

Exercise 1.8

Let $A = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$.

(a) Sketch A. Be careful to indicate which points lie in A.

(b) Determine whether A is closed for the Euclidean metric, d.

Closed sets for the discrete metric

We now look at closed sets when the discrete metric d_0 is used. We consider an arbitrary subset A of (X, d_0) and test whether d_0-convergent sequences of points from A must converge to a point in A; it turns out that *every subset of X is d_0-closed.*

Recall that for $x, y \in X$, $d_0(x, y) = 1$ unless $x = y$, in which case $d_0(x, y) = 0$.

Worked Exercise 1.11

Let X be a set. Show that every subset of X is d_0-closed.

Solution

Let $A \subseteq X$ be any subset and let (a_n) be a d_0-convergent sequence all of whose terms are in A. We must show that the limit of the sequence, a, say, also is a member of A.

Now, recall that a sequence in X is d_0-convergent if, and only if, it is eventually constant (see Worked Exercise 3.5 and Exercise 3.2 of Chapter 14) and so there is $N \in \mathbb{N}$ such that $a_n = a_N$ for $n \geq N$ and, in particular, a_N is the d_0-limit of the sequence (a_n). But by assumption $a_N \in A$ and so the limit of the sequence lies in A. Hence A is a d_0-closed set as required.

We saw earlier that intervals of the form $(-\infty, a)$, where $a \in \mathbb{R}$, and the set $\mathbb{Q}$ of rational numbers are not closed in the metric space $(\mathbb{R}, d^{(1)})$. However, it follows from the result of Worked Exercise 1.11 that each of these sets *is* closed in the space $(\mathbb{R}, d_0)$. Thus a set can be closed with respect to one metric but not with respect to another.

Closed sets in Cantor space

Recall from Chapter 15 that the Cantor space $\mathbf{C}$ consists of all infinite sequences of zeros and ones, and that we define the Cantor metric $d_{\mathbf{C}}$ by

$$d_{\mathbf{C}}(\mathbf{x},\mathbf{y}) = \begin{cases} 0, & \text{if } \mathbf{x} = \mathbf{y}, \\ 2^{-n}, & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ first differ at their } n\text{th term.} \end{cases}$$

In view of Proposition 1.3, we know that closed balls in Cantor space are $d_{\mathbf{C}}$-closed sets. But combining the results of Worked Exercise 2.4 and Exercise 2.3 of Chapter 15, we see that for $\mathbf{a} \in \mathbf{C}$ and $n \in \mathbb{N}$,

$$B_{d_{\mathbf{C}}}(\mathbf{a}, 2^{-n}) = B_{d_{\mathbf{C}}}[\mathbf{a}, 2^{-(n+1)}] = \{(a_1, a_2, \ldots, a_n, c_1, c_2, \ldots) : (c_i) \in \mathbf{C}\}.$$

Thus open balls in $\mathbf{C}$ are also $d_{\mathbf{C}}$-closed sets! This turns out to be a symptom of the fact that Cantor space is not connected – a topic that we take up in Book F, Chapter 21.

Closed sets in $(C[0,1], d_{\max})$

We give a couple of examples of closed subsets in $C[0,1]$.

Worked Exercise 1.12

Let $M \geq 0$ be given and let

$$A_M = \{f \in C[0,1] : |f(x)| \leq M \text{ for each } x \in [0,1]\},$$

so A_M consists of those continuous functions on $[0,1]$ whose values lie in the range $[-M, M]$.

Show that A_M is a closed subset of $(C[0,1], d_{\max})$.

Solution

Suppose that (f_n) is a $d_{\max}$-convergent sequence of functions in $C[0,1]$ with $f_n \in A_M$ for each n. We must show that the limit of the sequence, f, say, also lies in A_M. That is, for each $x \in [0,1]$, we must show $|f(x)| \leq M$. So suppose $x \in [0,1]$; then for each $n \in \mathbb{N}$ we can use the Triangle Inequality to find an upper bound for $|f(x)|$ involving f_n as follows:

$$|f(x)| = |f(x) - f_n(x) + f_n(x)| \leq |f(x) - f_n(x)| + |f_n(x)|.$$

But $|f(x) - f_n(x)| \leq d_{\max}(f_n, f)$ and $f_n \in A_M$, and so $|f_n(x)| \leq M$. Hence

$$|f(x)| \leq d_{\max}(f_n, f) + M \text{ for each } n \in \mathbb{N}.$$

But $(d_{\max}(f_n, f))$ is a null sequence and the value of $|f(x)|$ does not depend on n, and so we conclude that

$$|f(x)| \leq M.$$

Since x was an arbitrary real number in $[0,1]$, this implies that $f \in A_M$. We conclude that A_M is $d_{\max}$-closed.

An alternative way to show this would be to observe that $A_M = B_{d_{\max}}[g, M]$ where $g \in C[0,1]$ is the 'zero function' given by $g\colon [0,1] \to \mathbb{R}$, $g(x) = 0$, and so A_M is $d_{\max}$-closed by Proposition 1.3. We will see yet another (more flexible) way of obtaining this result in the next subsection.

Exercise 1.9

Let $M \in \mathbb{R}$ be given and let

$$B_M = \{f \in C[0,1] : f(0) = M\},$$

so B_M consists of those continuous functions on $[0,1]$ that take the value M at zero.

Show that B_M is a closed subset of $(C[0,1], d_{\max})$.

Exercise 1.10

Let $M \geq 0$ and $a \in [0,1]$ be given, and let

$$C = \{f \in C[0,1] : |f(a)| \leq M\},$$

so C consists of those continuous functions on $[0,1]$ that take a value in $[-M, M]$ at the point a.

Show that C is a closed subset of $(C[0,1], d_{\max})$.

1.2 The algebra of closed sets

We now have some examples of closed sets in several different kinds of metric space but, as you may have already concluded, it can be a bit tedious to verify directly from the definition of closed sets whether a given set is closed. In this subsection we look at how closed sets can be combined; in the same way that we use combination rules to combine known convergent sequences in $(\mathbb{R}, d^{(1)})$, this gives us a way to combine known closed sets to form new closed sets. In particular we investigate when unions and intersections of closed sets are also closed.

Before starting we record a useful result concerning convergent sequences in metric spaces.

Lemma 1.13

Let (X, d) be a metric space and $(a_n)_{n=1}^{\infty}$ be a sequence in X. If (a_n) is d-convergent, then any subsequence of (a_n) is also d-convergent and has the same limit as the original sequence.

Proof Let (a_n) be d-convergent with limit $a \in X$ and let $(a_{n_i})_{i=1}^{\infty}$ be a subsequence of (a_n). We must show that $(d(a_{n_i}, a))_{i=1}^{\infty}$ is a (real) null sequence.

Consider the real sequence (d_n) where $d_n = d(a_n, a)$; this is a real null sequence and the sequence $(d(a_{n_i}, a))_{i=1}^{\infty} = (d_{n_i})_{i=1}^{\infty}$ is a subsequence of it. Hence by Proposition 1.25 of Chapter 13, (d_{n_i}) is also convergent with limit 0. Thus $(d(a_{n_i}, a))_{i=1}^{\infty}$ is a (real) null sequence. ■

This result tells us that if, for a given sequence, we can find two subsequences that converge to different limits, then the original sequence must be divergent.

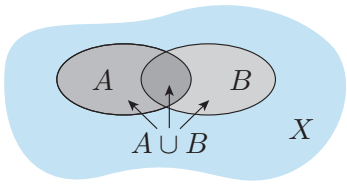


Figure 1.6 The union of A and B

We can now prove our first result concerning combining closed sets. It concerns the union of two sets, as shown in Figure 1.6. Recall that the union of two sets A and B, denoted $A \cup B$, is given by $A \cup B = \{x \in X:$ either $x \in A$ or $x \in B\}$. The 'or' in the definition of union is inclusive in that elements of the union of A and B can appear in both A and B.

Proposition 1.14

Let (X, d) be a metric space and let A and B be d-closed subsets of X. Then $A \cup B$ is also d-closed.

Proof Let (x_n) be a d-convergent sequence all of whose terms lie in $A \cup B$, and suppose it has limit x: we must show that $x \in A \cup B$. Then for at least one of A and B there is a subsequence of (x_n), $(x_{n_i})_{i=1}^{\infty}$, all of whose terms lie in it: suppose it is A. (So $x_{n_i} \in A$ for each $i \in \mathbb{N}$.) Then by Lemma 1.13, since (x_n) is convergent, so is the subsequence (x_{n_i}) and it has the same limit x. But $x_{n_i} \in A$ for each i and A is closed and so $x \in A$, which in particular means that $x \in A \cup B$ as required. ■

This result immediately tells us that the union of a finite number of closed sets is also closed. For a finite union of sets $A_1 \cup A_2 \cup \cdots \cup A_n$ can be written as the union of two sets $(A_1 \cup A_2 \cup \cdots \cup A_{n-1}) \cup A_n$ and we can use induction to deduce that if all the sets $A_1, \ldots, A_n$ are closed, then so is their union.

Thus, for example, a finite union of closed intervals is itself closed (with respect to the usual Euclidean metric).

Recall that a closed interval $[a, b] = B_d[\frac{1}{2}(a+b), \frac{1}{2}(b-a)]$ for the Euclidean metric d.

Worked Exercise 1.15

Let $a, b \in \mathbb{R}$ with $a \leq b$. Show that the complement of the open interval (a, b) is closed for the usual Euclidean metric.

Solution

The complement of the open interval (a, b) is

$$\mathbb{R} - (a, b) = (-\infty, a] \cup [b, \infty).$$

We saw in Worked Exercise 1.4 and Exercise 1.5 that the intervals $(-\infty, a]$ and $[b, \infty)$ are closed for the Euclidean metric on $\mathbb{R}$. Hence Proposition 1.14 implies that their union is also closed for the Euclidean metric.

Exercise 1.11

Let (X, d) be a metric space and let $x_1, x_2, \ldots, x_n \in X$. Show that the finite set $\{x_1, x_2, \ldots, x_n\}$ is d-closed.

Exercise 1.12

Let d denote the usual Euclidean metric on $\mathbb{R}$. Give examples of sets $A_1, A_2, A_3, \ldots$ in $\mathbb{R}$ such that A_n is d-closed for each n but $\cup_{n=1}^{\infty} A_n = \{a \in X : a \in A_n \text{ for some } n \in \mathbb{N}\}$ is not d-closed.

Hint: you may find it helpful to consider sets that each consist of a single term of an appropriately chosen convergent sequence.

Exercise 1.13

Let $K_0 = [0, 1]$ and suppose that for each $n \in \mathbb{N}$, K_n is obtained from the set K_{n-1} by replacing each interval I in K_{n-1} with two closed intervals of one-third the length of I, one starting at the left endpoint of I and the other ending at the right endpoint of I.

(a) Write down K_1 and K_2 as a union of intervals.

(b) Prove that K_n is closed for the Euclidean metric on $\mathbb{R}$ for each $n \in \mathbb{N}$.

The next result concerns the intersection of an arbitrary collection of closed sets, so before giving it, let us recall the definition of intersection. We already know what the intersection of two sets $A \cap B$ means – it consists of those points that are members of both A and B – but we are going to need to look at intersections of many more than two sets, so we should agree the terminology.

Definition 1.16 *Intersection*

Let $\{A_i : i \in I\}$ be a collection of subsets of a given set X.

The **intersection** of the collection of sets, written $\bigcap_{i\in I} A_i$ or $\cap\{A_i : i \in I\}$, is defined by

$$\bigcap_{i\in I} A_i = \{x \in X : x \in A_i \text{ for each } i \in I\}.$$

Thus the intersection consists of exactly those points that the sets A_i have in common.

The set I is called an **index set** for the collection.

Remarks

1. The index set can be finite or infinite and gives a way of labelling the sets so that we can refer to them directly.
2. For each $i \in I$, $\bigcap_{i\in I} A_i \subseteq A_i$. That is, the intersection of a collection of sets is always contained inside each of the sets in the collection.

Proposition 1.17

Let (X, d) be a metric space and let $\{A_i : i \in I\}$ be a collection of d-closed sets. Then $\bigcap_{i\in I} A_i$ is d-closed.

Proof Let (x_n) be a d-convergent sequence all of whose terms lie in $\bigcap_{i\in I} A_i$, and suppose that the limit of the sequence is x – we must show that for each $i \in I$, $x \in A_i$.

So fix $i \in I$. Since for each $n \in \mathbb{N}$, $x_n \in \bigcap_{i\in I} A_i$, it is certainly the case that $x_n \in A_i$ for each $n \in \mathbb{N}$.

But A_i is d-closed and (x_n) is a d-convergent sequence lying within A_i. Hence the limit of the sequence (x_n), x, also lies in A_i, as required. ■

We have shown that closed balls $B_d[x, r]$ are closed sets in a metric space (and that open balls, $B_d(x, r)$, need not be), but what about spheres?

Theorem 1.18

Let (X, d) be a metric space, let $a \in X$ and let $r \geq 0$. Then the sphere $S_d(a, r) = \{x \in X : d(a, x) = r\}$ is a d-closed set.

Exercise 1.14

Prove Theorem 1.18.

Example 1.19 *The middle-third Cantor set*

We now take a first look at an example of a closed set in the unit interval (with the usual Euclidean distance) that plays an important role in the final chapter of the module, Chapter 24.

As in Exercise 1.13, let $K_0 = [0, 1]$ and form K_1 from K_0 by removing the open middle-third of K_0 – the open interval $(\frac{1}{3}, \frac{2}{3})$, meaning that $K_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, the union of two (closed) intervals of length $\frac{1}{3}$. We continue in this way, constructing a set K_n for each $n \in \mathbb{N}$. Thus K_n is formed by removing the open middle-third of each interval in K_{n-1} and so K_n consists of 2^n closed intervals each with length $\frac{1}{3^n}$. Note that the sets K_n are the same as those in Exercise 1.13 and so are closed for the Euclidean metric.

We define the (middle-third) Cantor set K_C by $K_C = \bigcap_{n=1}^{\infty} K_n$. Then K_C is non-empty, since it contains the points 0 and 1, and, by Proposition 1.17, it is closed. However, notice that it does not contain any intervals because if x and y are in K_C with $x \neq y$ then, on choosing n so that $3^{-n} < |x - y|$, we see that x and y cannot lie in the same interval from K_n and so there is a point z lying between x and y that is not in K_C (Figure 1.7). Hence, in particular, if (x, y) is an (open) interval then there is $x < z < y$ with $z \notin K_C$. Thus K_C contains no intervals – it is an example of what is called a *totally disconnected set*. We investigate this and related ideas more fully in Chapter 21.

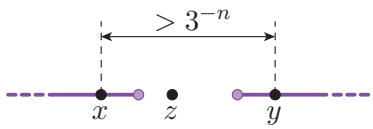


Figure 1.7 The point z lies between x and y but is not in K_C

Worked Exercise 1.20

Fix $M \geq 0$ and let

$$D = \{f \in C[0, 1] : |f(x)| \leq M \text{ for each } x \in [0, \tfrac{1}{2}]\}.$$

Show that D is $d_{\max}$-closed.

Solution

The result of Exercise 1.10 tells us that for each $a \in [0, \frac{1}{2}]$, the set C_a given by

$$C_a = \{f \in C[0, 1] : |f(a)| \leq M\}$$

is $d_{\max}$-closed. Now

$$D = \bigcap_{a \in [0, \frac{1}{2}]} C_a$$

and so is the intersection of a collection of $d_{\max}$-closed sets. Hence by Proposition 1.17, D is $d_{\max}$-closed.

Exercise 1.15

For $M \in \mathbb{N}$, let

$$F_M = \left\{ f \in C[0,1] : f\left(\tfrac{1}{M}\right) = 0 \right\}.$$

(a) Show that F_M is a closed set in $(C[0,1], d_{\max})$.

(b) Give an example of a function f lying in $\bigcap_{M\in\mathbb{N}} F_M$.

(c) Show that $\bigcap_{M\in\mathbb{N}} F_M$ is a closed set in $(C[0,1], d_{\max})$.

2 Open sets

In the last section we extended the idea of a closed interval in $\mathbb{R}$ to the notion of a closed set in a metric space. We also know about open intervals in $\mathbb{R}$ such as (a,b) for $a < b$, and in this section we see how we can define a more general notion of being 'open' in a metric space.

Our starting point is based on some observations made in Worked Exercise 1.15 and Exercise 1.5 – namely that the 'open' intervals (a,b) and $(-\infty, a)$ both have complements that are closed sets in $\mathbb{R}$ with the usual Euclidean metric. This leads to an indirect way of defining the notion of 'open' for a general metric space in terms of it being the complement of a closed set.

Definition 2.1 *Open set*

Let (X,d) be a metric space and let $A \subseteq X$. Then A is said to be an **open set for the metric** d if the complement of A in X, $X - A$, is a d-closed set.

Remarks

1. When we wish to emphasise the particular metric d on X, we say that A is a d-**open set** or that the set A is **open** in (X,d).
2. Since the complements of intervals of the form (a,b) and $(-\infty,a)$ are closed for the usual Euclidean metric on $\mathbb{R}$, open intervals in $\mathbb{R}$ are open for the Euclidean metric. (Intervals of the form (a,∞) also have a closed complement.)

As we did with closed sets, we start by identifying some sets that are always open in a metric space – the proof of the following lemma makes use of Lemma 1.2.

Lemma 2.2

Let (X, d) be a metric space. The following subsets of X are d-open:

1. the empty set, $\emptyset$
2. the whole space, X.

Proof

1. The complement of the empty set in X is $X - \emptyset$, which equals X. From Lemma 1.2, we know that X is d-closed and so $\emptyset$ is d-open.
2. The complement of X in X is $X - X$, which equals the empty set $\emptyset$. From Lemma 1.2, we know that $\emptyset$ is d-closed and so X is d-open. ■

Thus the sets $\emptyset$ and X are both open and closed in any metric space (X, d). Later in this chapter we will see further examples of sets that are both open and closed in a particular metric space, $(\mathbf{C}, d_{\mathbf{C}})$.

We now look at open balls, $B_d(x, r)$, in a metric space and confirm that they are, indeed, d-open sets!

Proposition 2.3

Let (X, d) be a metric space, let $a \in X$ and let $r > 0$. Then the (open) ball $B_d(a, r) = \{x \in X : d(x, a) < r\}$ is a d-open set.

Proof In order to show that the ball $B_d(a, r)$ is open, we must show that $X - B_d(a, r)$ is d-closed. But, by the result of Worked Exercise 1.6, $X - B_d(a, r)$ is d-closed. Hence $B_d(a, r)$ is d-open as required. ■

2.1 Examples of open sets

We can take the complements of the examples of closed sets in various metric spaces that we found in Subsection 1.1 to obtain examples of open sets.

Example 2.4

Consider the plane $\mathbb{R}^2$ with the usual Euclidean metric d. Then the result of Worked Exercise 1.9 tells us that the complement of $H = [0, 1] \times [0, 3]$ is a d-open set. That is,

$$\begin{aligned}\mathbb{R}^2 - H &= \mathbb{R}^2 - \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq 3\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0 \text{ or } x_1 > 1 \text{ or } x_2 < 0 \text{ or } x_2 > 3\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 1\} \\ &\qquad \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_2 < 0\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 3\}\end{aligned}$$

is open for the Euclidean metric on the plane.

Similarly, Exercise 1.8 tells us that the complement of $A = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$ is an open set for the Euclidean metric on the plane. Thus

$$\begin{aligned}\mathbb{R}^2 - A &= \mathbb{R}^2 - \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 1\}\end{aligned}$$

is open.

Example 2.5

The situation for the discrete metric is interesting. We saw in Worked Exercise 1.11 that if a set X is given the discrete metric, d_0, then every subset of X is d_0-closed. This in turn means that every subset of X is also d_0-open: for if $A \subseteq X$, then $X - A$ is d_0-closed and so A must be open. Thus the discrete metric on a set X gives an example of a metric space where every subset is both open and closed!

Example 2.6

We saw several examples of closed sets for $(C[0,1], d_{\max})$ in Subsection 1.1 and, by taking complements, we immediately obtain examples of open sets. For example, in Exercise 1.9, we saw that B_M given by $B_M = \{f \in C[0,1] : f(0) = M\}$ for some $M \in \mathbb{R}$ is a $d_{\max}$-closed set. Hence for each $M \in \mathbb{R}$, $C[0,1] - B_M = \{f \in C[0,1] : f(0) \neq M\}$ is a $d_{\max}$-open set.

Exercise 2.1

Let $a \in \mathbb{R}$. Show that the interval (a, ∞) is open in $\mathbb{R}$ for the usual Euclidean metric and explain why $[a, \infty)$ is not.

Exercise 2.2

Let $a < b$. Explain why the sets $(a, b]$ and $[a, b]$ are not open for the Euclidean metric.

Exercise 2.2 highlights an important point: sets in a metric space need not be open or closed – the interval $(a, b]$ is neither open nor closed for the usual Euclidean metric.

Let us consider briefly the metric space $(C[0,1], d_{\max})$. Open sets in this space are more difficult to visualise than those in Euclidean spaces.

Exercise 2.3

Let $M \geq 0$ be given and let

$$C_M = \{f \in C[0,1] : \text{there is } x \in [0,1] \text{ for which } |f(x)| > M\},$$

so C_M consists of those continuous functions on $[0,1]$ that take a value outside the range $[-M, M]$ for some point in $[0,1]$.

Show that C_M is an open subset of $(C[0,1], d_{\max})$.

Working directly from the definition of open sets requires us to look at the complement of the set in question and show that this is closed for the metric. This is a rather indirect approach and it would be helpful if we had a way of showing a set is open without having to resort to looking at the complement each time. A clue towards doing this is provided by the following observation concerning open balls in a metric space.

Theorem 2.7

Let (X, d) be a metric space, let $a \in X$ and $r > 0$, and consider the open ball $B_d(a, r)$. Then, whenever $x \in B_d(a, r)$, we can find an $s > 0$ such that

$$B_d(x, s) \subseteq B_d(a, r).$$

We prove this shortly.

Example 2.8

Consider the open ball $B_{d^{(1)}}(2, 1)$ in $(\mathbb{R}, d^{(1)})$, that is, the open interval $(1, 3)$. Given any $x \in (1, 3)$, we can find $B_{d^{(1)}}(x, s)$; that is, the open interval $(x - s, x + s)$ such that

$$(x - s, x + s) \subseteq (1, 3).$$

For instance, if $x = 2.99$, then we may choose $s = 0.005$, since $(2.985, 2.995) \subseteq (1, 3)$.

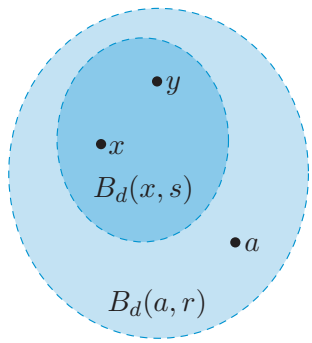


Figure 2.1
$B_d(x, s) \subseteq B_d(a, r)$

The properties of a metric space were first listed in Chapter 14, Definition 2.1.

We now prove Theorem 2.7.

Proof of Theorem 2.7 Let $x \in B_d(a, r)$, so $d(a, x) < r$. Now let $s = r - d(a, x) > 0$. Then, for $y \in B_d(x, s)$, we may use property (M3) (the Triangle Inequality) for d to conclude that

$$d(a, y) \leq d(a, x) + d(x, y) < d(a, x) + s = r.$$

That is, $y \in B_d(a, r)$. Thus $B_d(x, s) \subseteq B_d(a, r)$, as shown in Figure 2.1. ■

The closer to the boundary of the ball $B_d(a, r)$ that x lies, the smaller the radius s may be: the crucial point is that *there is always room enough*.

This observation about open balls extends to give us a useful characterisation of open sets.

Theorem 2.9

Let (X, d) be a metric space and let A be a subset of X. Then A is d-open if, and only if, for each $a \in A$ there is $r > 0$ so that $B_d(a, r) \subseteq A$.

Proof *If:*

We first suppose that for each $a \in A$, there is $r > 0$ so that $B_d(a, r) \subseteq A$. Our goal is to show that the complement of A, $X - A$, is d-closed.

So let (x_n) be a d-convergent sequence with limit x for which $x_n \in X - A$ for each n. If we show that $x \in X - A$, then $X - A$ is d-closed. So suppose instead that $x \notin X - A$, that is, $x \in A$.

Then by our hypothesis there is $s > 0$ so that $B_d(x, s) \subseteq A$. Since by Proposition 2.3 $B_d(x, s)$ is d-open, its complement, C, say, is d-closed. Now $(X - A) \subseteq C$ and so $x_n \in C$ for each $n \in \mathbb{N}$. But since C is d-closed and each of the terms $x_n \in C$, it follows that $x \in C$. But this is impossible because $C \cap B_d(x, s) = \emptyset$. Thus our assumption that $x \notin X - A$ is false and so $x \in X - A$. This shows that $X - A$ is a d-closed set and so, by definition, A is a d-open set.

Only if:

We now suppose that A is an open set and show that for each $a \in A$ there is $r > 0$ so that $B_d(a, r) \subseteq A$.

Let $B = X - A$ be the complement of A. Then since A is d-open, B is a d-closed set. If the claimed conclusion does not hold, then there is $a \in A$ such that for each $r > 0$, $B_d(a, r) \cap B \neq \emptyset$. In particular, for this choice of $a \in A$ and each $n \in \mathbb{N}$ we can choose $a_n \in B_d(a, \frac{1}{n}) \cap B$.

Thus, in particular, $a_n \in B$ for each $n \in \mathbb{N}$ and $d(a, a_n) \leq \frac{1}{n}$. This means that $(d(a, a_n))$ is a null sequence and hence that (a_n) is d-convergent with limit a.

But B is d-closed and (a_n) is a d-convergent sequence and so it must be the case that $a \in B$, which is a contradiction.

Thus no such a can exist and A has the claimed property. ■

Worked Exercise 2.10

Show that

$$S = (-1, 1) \times (-1, 1)$$

is open in $\mathbb{R}^2$ for the Euclidean metric.

Solution

There are two possible ways to approach this: either show that the complement of S is a closed set or use Theorem 2.9. We use Theorem 2.9 and show that for each $\mathbf{a} \in S$, we can find an $r > 0$ so that $B_d(\mathbf{a}, r) \subseteq S$ (where d denotes the Euclidean metric for the plane).

So let $\mathbf{a} = (a_1, a_2) \in S$. Then both $|a_1| < 1$ and $|a_2| < 1$.

Let $r = \min\{1 - |a_1|, 1 - |a_2|\}$, so that r is the minimum distance that the point $\mathbf{a}$ is from the 'boundary' of S (see Figure 2.2).

We claim that $B_d(\mathbf{a}, r) \subseteq S$. To show this, suppose $\mathbf{x} = (x_1, x_2) \in B_d(\mathbf{a}, r)$ – we must show that $\mathbf{x} \in S$. Now by the Triangle Inequality (for modulus), the fact that $|x_1 - a_1| \leq d(\mathbf{x}, \mathbf{a}) < r$ and the definition of r, we can find an upper bound for $|x_1|$:

$$\begin{aligned} |x_1| &= |x_1 - a_1 + a_1| \leq |x_1 - a_1| + |a_1| \\ &< r + |a_1| \leq (1 - |a_1|) + |a_1| = 1. \end{aligned}$$

Similarly, we can find an upper bound for $|x_2|$:

$$\begin{aligned} |x_2| &= |x_2 - a_2 + a_2| \leq |x_2 - a_2| + |a_2| \\ &< r + |a_2| \leq (1 - |a_2|) + |a_2| = 1. \end{aligned}$$

Thus both $|x_1| < 1$ and $|x_2| < 1$ and so $\mathbf{x} \in S$. We conclude that S is an open set.

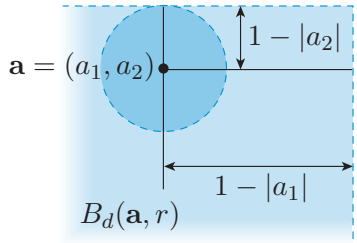


Figure 2.2 Minimum distance of $\mathbf{a}$ from the boundary of S

Worked Exercise 2.11

Let c be a point in $[0, 1]$, and consider the set A_c of all functions $f \in C[0, 1]$ whose value $f(c)$ at c lies in the real open interval $(-1, 1)$. If we think of $(-1, 1)$ as a vertical 'gate' sited at $x = c$, then f belongs to A_c only if it passes through the gate:

$$A_c = \{f \in C[0, 1] : |f(c)| < 1\}.$$

Figure 2.3 shows a function $g \in A_c$ and a function $h \notin A_c$.

Show that A_c is an open set in $(C[0, 1], d_{\max})$.

Solution

We show that for any $g \in A_c$ there is $r > 0$ so that $B_{d_{\max}}(g, r) \subseteq A_c$, which by Theorem 2.9 implies that A_c is $d_{\max}$-open.

Since $g(c) \in (-1, 1)$, we can find $r > 0$ with $r < 1 - |g(c)|$. Then

$$(g(c) - r, g(c) + r) \subseteq (-1, 1).$$

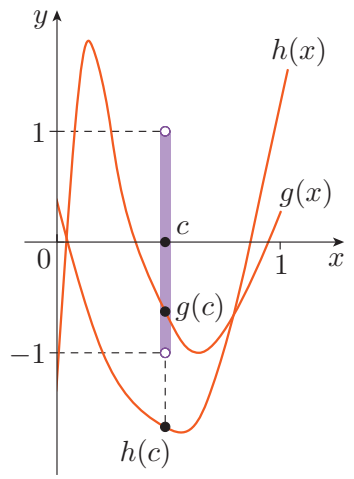


Figure 2.3 The function g lies in the set A_c of Worked Exercise 2.11 whereas the function h does not

With this value of r, consider the open ball $B_{d_{\max}}(g, r)$. If $h \in B_{d_{\max}}(g, r)$, then $|h(x) - g(x)| < r$ for each $x \in [0, 1]$, so that in particular $|h(c) - g(c)| < r$. Thus

$$-1 < g(c) - r < h(c) < g(c) + r < 1,$$

which shows that $h \in A_c$. Thus if $h \in B_{d_{\max}}(g, r)$, then $h \in A_c$ and so $B_{d_{\max}}(g, r) \subseteq A_c$.

Since g is an arbitrary point in A_c, it follows from Theorem 2.9 that A_c is an open subset of $(C[0, 1], d_{\max})$.

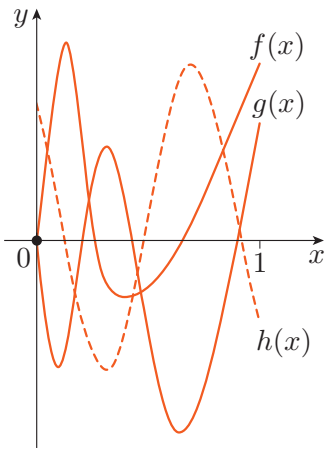


Figure 2.4 The functions f and g lie in S, but the function h does not

It is not too difficult to find examples of sets in $(C[0, 1], d_{\max})$ that are not open. For example, consider the set S of all functions whose graphs pass through the origin:

$$S = \{f \in C[0, 1] : f(0) = 0\}.$$

Figure 2.4 shows two functions f and g in S, and a function h not in S.

Exercise 2.4

Show that the set S is not open.

We saw when looking at examples of closed sets in Subsection 1.1 that in Cantor space $(\mathbf{C}, d_{\mathbf{C}})$ the open balls $B_{d_{\mathbf{C}}}(\mathbf{a}, r)$ are $d_{\mathbf{C}}$-closed sets. There is a symmetric result concerning closed balls in Cantor space.

Exercise 2.5

Let $(\mathbf{C}, d_{\mathbf{C}})$ be the Cantor space with the Cantor metric. Show that for $\mathbf{a} \in \mathbf{C}$ and $n \in \mathbb{N}$, the closed ball $B_{d_{\mathbf{C}}}[\mathbf{a}, 2^{-n}]$ is a $d_{\mathbf{C}}$-open set.

In Chapter 14 we observed that a sequence (a_n) in X converges to $a \in X$ if, for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $a_n \in B_d(a, \varepsilon)$ whenever $n > N$. The next exercise asks you to interpret this result in a slightly different way.

Exercise 2.6

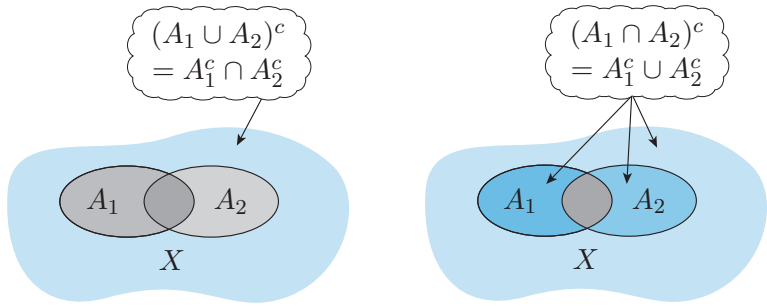
Let (a_n) be a sequence in the metric space (X, d) that does not converge to the point $a \in X$. Show that there is an open ball $B_d(a, \varepsilon)$ (for some $\varepsilon > 0$) such that there is a subsequence (a_{m_n}) with $a_{m_n} \notin B_d(a, \varepsilon)$ for each $n \in \mathbb{N}$.

2.2 The algebra of open sets

In the same way as we investigated how closed sets could be combined through union and intersection in Subsection 1.2, we now look at how open sets can be combined. Given that open sets are defined in terms of the complements of closed sets, one would expect that the rules for their combination can be derived from those for closed sets – and indeed, this is the case. To do this, we make use of *De Morgan's Laws*, which give identities useful for calculating complements of unions and intersections of sets. In the simplest case, when we have just two sets A_1 and A_2 lying within a 'universal set' X, these laws take the following form:

$$X - (A_1 \cup A_2) = (A_1 \cup A_2)^c = A_1^c \cap A_2^c = (X - A_1) \cap (X - A_2);$$
$$X - (A_1 \cap A_2) = (A_1 \cap A_2)^c = A_1^c \cup A_2^c = (X - A_1) \cup (X - A_2).$$

These are illustrated in Figure 2.5.



(a) (b)

Figure 2.5 (a) $X - (A_1 \cup A_2)$; (b) $X - (A_1 \cap A_2)$

Augustus De Morgan (1806–1871) contributed to set theory and mathematical logic, and was a distinguished populariser of mathematics. He was a founding member of the London Mathematical Society, which today, despite its name, is a leading UK-wide learned society for mathematics.

Exercise 2.7

For $X = \{1, 2, \ldots, 12\}$, let $A_1 = \{x \in X : x \text{ is even}\}$ and let $A_2 = \{x \in X : x \text{ is divisible by } 3\}$. Verify De Morgan's Laws for the sets A_1 and A_2.

The most general form of De Morgan's Laws applies to arbitrary families of subsets of a set X.

Theorem 2.12 *De Morgan's Laws*

Let $\{A_i : i \in I\}$ be a family of subsets of X. Then:

First Law

$$X - \left(\bigcup_{i\in I} A_i\right) = \left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c = \bigcap_{i\in I}(X - A_i);$$

Second Law

$$X - \left(\bigcap_{i\in I} A_i\right) = \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c = \bigcup_{i\in I}(X - A_i).$$

Although it is not difficult, we do not prove this result.

De Morgan's First Law can be remembered as 'the complement of the union is (identical to) the intersection of the complements' and De Morgan's Second Law as 'the complement of the intersection is the union of the complements'.

Exercise 2.8

Write out De Morgan's Laws when the index set is $I = \{1, 2, 3\}$.

We now use De Morgan's Laws together with Propositions 1.14 and 1.17 to investigate how open sets are combined.

Theorem 2.13

In a metric space, the intersection of any two open sets is an open set.

Proof Let (X, d) be a metric space, and suppose that U and V are d-open subsets of X. Then, by definition, $X - U$ and $X - V$ are d-closed sets.

Consider $U \cap V$: to show that this is a d-open set, it is enough to show that $(U \cap V)^c = X - (U \cap V)$ is d-closed.

However, by De Morgan's Second Law,

$$X - (U \cap V) = (X - U) \cup (X - V)$$

and both $X - U$ and $X - V$ are d-closed. Hence, by Proposition 1.14, so is their union, implying that $U \cap V$ is d-open as required. ■

Remarks

1. If U_1, U_2, U_3 are open in the metric space (X, d), then

   $$U_1 \cap U_2 \cap U_3 = (U_1 \cap U_2) \cap U_3$$

   and we may use Theorem 2.13 first to deduce that $U_1 \cap U_2$ is open and then to deduce that $(U_1 \cap U_2) \cap U_3$ is open. Thus, the intersection of any three open sets is open.

   This method extends to any *finite* intersection of open sets and allows us to conclude that if $U_1, U_2, \ldots, U_n$ are open sets in a metric space (X, d), then so is

   $$U = U_1 \cap U_2 \cap \cdots \cap U_n.$$

2. However, this result does *not* extend to the intersection of *infinitely many* open sets in a metric space. For example, consider the following collection of open intervals in $\mathbb{R}$ with its usual metric $d(x, y) = |x - y|$:

   $$A_1 = (-1, 1), A_2 = (-\tfrac{1}{2}, \tfrac{1}{2}), A_3 = (-\tfrac{1}{3}, \tfrac{1}{3}), \ldots, A_n = (-\tfrac{1}{n}, \tfrac{1}{n}), \ldots,$$

   illustrated in Figure 2.6. Each set A_n (being an open interval) is open, but

   $$\cap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \cdots$$

   is the set of real numbers lying in *all* the intervals $(-\frac{1}{n}, \frac{1}{n})$ for all $n \in \mathbb{N}$, which is the singleton set $\{0\}$. But $\{0\}$ is not an open set, since any d-open ball about 0 contains points other than 0.

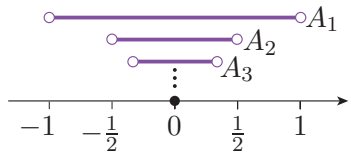


Figure 2.6 $A_n = (-\frac{1}{n}, \frac{1}{n})$

Since Proposition 1.17 tells us that arbitrary intersections of closed sets are closed and De Morgan's Laws have the effect of turning complements of intersections into unions of complements, we would expect that the union of *any* collection of open sets is an open set, irrespective of whether the collection is finite or infinite.

We first remind ourselves of the meaning of the union of a collection of sets.

Definition 2.14 *Union*

Let $\{A_i : i \in I\}$ be a collection of subsets of a given set X.

The **union** of the collection of sets, written $\bigcup_{i \in I} A_i$ or $\cup\{A_i : i \in I\}$, is defined by

$$\bigcup_{i \in I} A_i = \{x \in X : \text{there is } i \in I \text{ for which } x \in A_i\}.$$

Thus the union consists of exactly those points that appear in at least one of the sets A_i.

We can now state our result concerning unions of open sets.

Theorem 2.15

In a metric space, the union of any collection of open sets is open.

Exercise 2.9

Prove Theorem 2.15.

Any *finite* non-empty set of real numbers $A = \{a_1, a_2, \ldots, a_n\}$ is not open in $(\mathbb{R}, d^{(1)})$: in any open ball centred on a point in A, there are points not in A. Similarly, if (a_n) is a strictly increasing sequence of real numbers,

$$a_1 < a_2 < a_3 < \cdots,$$

then the set

$$A = \{a_n : n \in \mathbb{N}\}$$

cannot be open in $(\mathbb{R}, d^{(1)})$. To see this, consider any point a_k of A: for any $r > 0$, $B_{d^{(1)}}(a_k, r)$ contains points not in A. Thus, for example, the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers is not open in $(\mathbb{R}, d^{(1)})$.

Worked Exercise 2.16

Show that the set $\mathbb{Q}$ of rational numbers is not open in $(\mathbb{R}, d^{(1)})$.

Solution

Let $x \in \mathbb{Q}$. We show that for each $r > 0$, there is a point $y \in B_{d^{(1)}}(x, r)$ such that $y \notin \mathbb{Q}$ – that is, each open ball centred on a rational number must contain an irrational number.

So, let $r > 0$ and consider $y = x + \frac{\sqrt{2}}{k}$, where $k \in \mathbb{N}$ is chosen so that $\frac{\sqrt{2}}{k} < r$. It follows that $y \in B_{d^{(1)}}(x, r)$. We now need to verify that y is irrational. Suppose $y = x + \frac{\sqrt{2}}{k}$ is rational. Then, on rearranging, we obtain

$$k(y - x) = \sqrt{2}.$$

This implies that $\sqrt{2}$ is rational, which is false, so our assumption must be false. Thus, y is irrational and so is not in $\mathbb{Q}$.

Hence $B_{d^{(1)}}(x, r) \not\subseteq \mathbb{Q}$ and it follows, since r was arbitrary, that $\mathbb{Q}$ is not open.

One consequence of Theorem 2.15 is that if we can write a given set as a union of open sets, then we immediately know that it is open and we do not have to find open balls around each point to verify this. For example, let

$$J = \{(x, 0) \in \mathbb{R}^2 : -2 \leq x \leq 2\},$$

and let

$$U = \{\mathbf{u} \in \mathbb{R}^2 : d^{(2)}(\mathbf{u}, \mathbf{j}) < 1 \text{ for some } \mathbf{j} \in J\}.$$

Thus U is a rectangle with 'rounded ends' (Figure 2.7), and it may be intuitively clear to you that U is open. We could construct a proof of this from the definition of an open set, but there is no need: because U has been defined as the union of the infinite collection of open balls $B_{d^{(2)}}((x, 0), 1)$ where $-2 \leq x \leq 2$, we can deduce immediately from Theorem 2.15 that U is open.

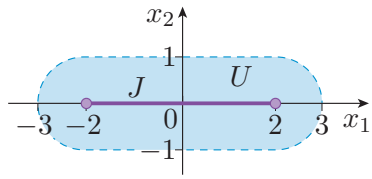


Figure 2.7 Representation of U

This principle is also effective for more abstract metric spaces, such as $(C[0, 1], d_{\max})$. For example, we saw in Worked Exercise 2.11 that for each $c \in [0, 1]$, the set $A_c = \{f \in C[0, 1] : |f(c)| < 1\}$ is $d_{\max}$-open. Hence we can deduce immediately from Theorem 2.15 that the set

$$A = \{f \in C[0, 1] : \text{there is a } c \in [0, 1] \text{ for which } |f(c)| < 1\}$$

is a $d_{\max}$-open set, since A is the union of the infinite collection of sets A_c for $c \in [0, 1]$.

Exercise 2.10

Show that the following are open sets in the metric space $(\mathbb{R}^2, d)$ when d is the usual Euclidean metric for the plane.

(a) $U = \{(x, y) \in \mathbb{R}^2 : y > 1\}$

(b) $V = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + (y - 1)^2 < 1 \text{ and } (x - 2)^2 + (y - 2)^2 < 1\}$

(c) The set W of points $(x, y) \in \mathbb{R}^2$ such that there is a point $(a, b) \in \mathbb{R}^2$ with $a^2 + b^2 = 1$ for which

$$(x - a)^2 + (y - b)^2 < \tfrac{1}{4}.$$

We finish this section by summarising various ways of showing that a given set is open or closed or neither.

Strategy: to determine whether sets are open or closed

Let (X, d) be a metric space and let $A \subseteq X$. We list the various techniques that you have seen for showing whether A is open, closed or neither. Remember that A can be both open and closed or could be neither.

- To show A is closed:
  - show that the limits of every convergent sequence with terms all lying in A lie in A; or
  - show that A is the union of a finite number of known closed sets; or
  - show that A is the intersection of an arbitrary collection of known closed sets; or
  - show that the complement of A is open.
- To show A is open:
  - show that the complement of A (in X) is closed; or
  - show that for each $a \in A$, there is $r > 0$ so that $B_d(a, r) \subseteq A$; or
  - show that A is the intersection of a finite number of known open sets; or
  - show that A is the union of an arbitrary collection of known open sets.
- To show A is not closed:
  - show that there is a convergent sequence (a_n) with $a_n \in A$ for each n and $\lim_{n\to\infty} x_n \notin A$; or
  - show that the complement of A is not open.
- To show A is not open:
  - show that there is a point $a \in A$ with no open ball $B_d(a, \varepsilon) \subseteq A$ – that is, show that for each open ball $B_d(a, \varepsilon)$, we have $B_d(a, \varepsilon) \cap (X - A) \neq \emptyset$; or
  - show that the complement of A is not closed.

3 Closure

Given a metric space (X, d) we have identified two particular important types of sets lying in X: the closed sets, defined via sequences, and the open sets, which are sets that are complements of closed sets. However, many subsets of metric spaces are neither open nor closed and it would be helpful to have a natural way of associating an open and a closed set with any given subset of a metric space – this is what we investigate in this section.

We have seen that an interval such as $(0, 1]$ is an example of a subset of $\mathbb{R}$ that is neither open nor closed for the Euclidean metric; we now use it to investigate how we could associate a natural open and closed set with it.

First, let us consider how to associate a reasonable closed set with $(0, 1]$. With luck, one such set may already have sprung to your mind, namely the interval $[0, 1]$. This is certainly a closed set for the Euclidean metric, and it is not so very different from our starting set A in that it only contains a single extra point, 0. Similarly, if we try to associate a natural open set with the set A, then the open interval $(0, 1)$ springs to mind.

Exercise 3.1

For each of the following subsets of $\mathbb{R}$, state the open and closed sets that you would naturally associate with them (for the usual Euclidean metric):

(a) $[0, 1)$; (b) $[0, 1]$; (c) $(0, 1)$; (d) $\{0\}$.

In the preceding exercise, we saw that a natural associated closed set always contained the starting set and was the 'smallest' closed set containing the starting set. Similarly we saw that a natural associated open set is the 'largest' open set that lies inside the starting set. This is not a coincidence. Suppose that we have been given a set A in a metric space (X, d) and we wish to find a closed set that contains it. There is always at least one such closed set, namely the space X itself. Now suppose that we have a family of closed sets $\{E_i : i \in I\}$, each of which contains A. Then:

1. $A \subseteq \bigcap_{i\in I} E_i$, since $A \subseteq E_i$ for each $i \in I$.
2. Proposition 1.17 tells us that $\bigcap_{i\in I} E_i$ is also a closed set.

So, given any collection of closed sets containing A, their intersection is also a closed set that contains A.

Consider now the collection

$$\{E \subseteq X : A \subseteq E \text{ and } E \text{ is } d\text{-closed in } X\}.$$

Then, in view of the preceding, the set F that is defined by

$$F = \bigcap \{E \subseteq X : A \subseteq E \text{ and } E \text{ is } d\text{-closed in } X\}$$

is a closed set that contains A. Moreover if E is a closed set that contains A, then E occurs in the intersection defining F and so $A \subseteq F \subseteq E$. In

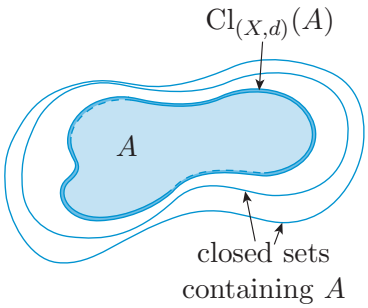


Figure 3.1 The closure of A

other words, F is the smallest closed set that contains A. This is our sought-after definition of the *closure* of a set A.

Definition 3.1 *Closure of a set*

Let (X, d) be a metric space and let $A \subseteq X$.

We define the **closure of A in (X, d)**, $\mathrm{Cl}_{(X,d)}(A)$, to be the smallest d-closed set in X that contains A (Figure 3.1). That is,

$$\mathrm{Cl}_{(X,d)}(A) = \bigcap \{E \subseteq X : A \subseteq E \text{ and } E \text{ is } d\text{-closed in } X\}.$$

Remarks

1. When it is clear from the context we usually just write $\mathrm{Cl}(A)$ and suppress mention of the metric space (X, d).
2. The closure of a set is closed.
3. It is always the case that $A \subseteq \mathrm{Cl}_{(X,d)}(A)$.
4. If F is a d-closed set and $A \subseteq F$, then $\mathrm{Cl}_{(X,d)}(A) \subseteq F$.

We can go through a similar process to find a suitable definition for the 'largest open set' inside a given set and we do this in Section 4.

The next couple of exercises give you the opportunity to explore closure.

Exercise 3.2

Show that for every metric space (X, d),

$$\mathrm{Cl}_{(X,d)}(\emptyset) = \emptyset \text{ and } \mathrm{Cl}_{(X,d)}(X) = X.$$

Exercise 3.3

Determine $\mathrm{Cl}_{(\mathbb{R},d^{(1)})}((0, 1))$.

3.1 Elementary properties of closures

The following simple properties of closures, which follow immediately from the definition, are very useful.

Theorem 3.2

Let (X, d) be a metric space and let $A, B \subseteq X$ with $A \subseteq B$. Then

$$\mathrm{Cl}_{(X,d)}(A) \subseteq \mathrm{Cl}_{(X,d)}(B).$$

Proof Suppose that $A \subseteq B$. Then certainly $B \subseteq \mathrm{Cl}(B)$ and so $\mathrm{Cl}(B)$ is a closed set that contains A. Hence

$$A \subseteq \mathrm{Cl}(A) = \bigcap \{E \subseteq X : A \subseteq E \text{ and } E \text{ is } d\text{-closed in } X\} \subseteq \mathrm{Cl}(B),$$

as required. ■

We now prove that closed sets are precisely those sets that are equal to their closure.

Theorem 3.3

Let (X, d) be a metric space and let $A \subseteq X$. Then A is closed if, and only if, $\mathrm{Cl}_{(X,d)}(A) = A$.

Proof We already know that, by its definition, $\mathrm{Cl}(A)$ is a closed set. So certainly if $\mathrm{Cl}(A) = A$, then A is also a closed set.

Now suppose that A is closed. We must show that $\mathrm{Cl}(A) = A$. We know that it is always the case that $A \subseteq \mathrm{Cl}(A)$, and so it is enough to show that $\mathrm{Cl}(A) \subseteq A$. But if A is a closed set, then it is a closed set that contains A and so

$$A \subseteq \bigcap \{E \subseteq X : A \subseteq E \text{ and } E \text{ is closed}\} \subseteq A.$$

Thus we conclude that $\mathrm{Cl}(A) = A$. ■

Remarks

1. A consequence of this theorem is that if A is not a closed set, then A is a *proper* subset of $\mathrm{Cl}(A)$: $\mathrm{Cl}(A) - A \neq \emptyset$.
2. Since $\mathrm{Cl}(A)$ is a closed set,
$$\mathrm{Cl}(\mathrm{Cl}(A)) = \mathrm{Cl}(A).$$

Worked Exercise 3.4

Find the closure of

$$A = \{f \in C[0,1] : f(\tfrac{1}{n}) = 0 \text{ for each } n \in \mathbb{N}\}$$

in the metric space $(C[0,1], d_{\max})$.

Solution

We saw in part (c) of Exercise 1.15 that A is closed for $d_{\max}$ and so, by Theorem 3.3, $\mathrm{Cl}(A) = A$.

Exercise 3.4

Let $I \subseteq [0,1]$ be a non-empty set. Find the closure of

$$A = \{f \in C[0,1] : f(x) = 0 \text{ for } x \in I\}$$

in the metric space $(C[0,1], d_{\max})$.

3.2 Further properties of closures

See Theorem 3.2.

Suppose that we have two subsets of a metric space. We have already observed that if $A \subseteq B$ then $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$. But suppose we do not know whether one set is contained in the other. What is the relationship between the closure of their union and the union of their closures? What happens when we look at the intersection of their closures? The next theorem provides some of the answers.

Theorem 3.5

Let (X, d) be a metric space and let $A, B \subseteq X$. Then:

(a) $\mathrm{Cl}_{(X,d)}(A \cup B) = \mathrm{Cl}_{(X,d)}(A) \cup \mathrm{Cl}_{(X,d)}(B)$

(b) $\mathrm{Cl}_{(X,d)}(A \cap B) \subseteq \mathrm{Cl}_{(X,d)}(A) \cap \mathrm{Cl}_{(X,d)}(B)$.

Proof We first prove part (a) of the theorem.

We start by showing that

$$\mathrm{Cl}(A \cup B) \subseteq \mathrm{Cl}(A) \cup \mathrm{Cl}(B).$$

We know that $A \subseteq \mathrm{Cl}(A)$ and $B \subseteq \mathrm{Cl}(B)$ and so

$$A \cup B \subseteq \mathrm{Cl}(A) \cup \mathrm{Cl}(B).$$

In particular, since $\mathrm{Cl}(A) \cup \mathrm{Cl}(B)$ is a closed set (being the union of two closed sets) that contains $A \cup B$, it follows that

$$\mathrm{Cl}(A \cup B) \subseteq \mathrm{Cl}(A) \cup \mathrm{Cl}(B).$$

It remains only to show that

$$\mathrm{Cl}(A) \cup \mathrm{Cl}(B) \subseteq \mathrm{Cl}(A \cup B).$$

To do this, notice that $A \subseteq A \cup B \subseteq \mathrm{Cl}(A \cup B)$ and similarly $B \subseteq A \cup B \subseteq \mathrm{Cl}(A \cup B)$. Hence

$$\mathrm{Cl}(A) \subseteq \mathrm{Cl}(\mathrm{Cl}(A \cup B)) = \mathrm{Cl}(A \cup B), \text{ by Theorem 3.3,}$$

and

$$\mathrm{Cl}(B) \subseteq \mathrm{Cl}(\mathrm{Cl}(A \cup B)) = \mathrm{Cl}(A \cup B).$$

Therefore

$$\mathrm{Cl}(A) \cup \mathrm{Cl}(B) \subseteq \mathrm{Cl}(A \cup B).$$

Thus

$$\mathrm{Cl}(A) \cup \mathrm{Cl}(B) = \mathrm{Cl}(A \cup B),$$

as required.

You are asked to verify part (b) in Exercise 3.5. ■

Exercise 3.5

Let (X, d) be a metric space and let $A, B \subseteq X$. Prove that

$$\mathrm{Cl}_{(X,d)}(A \cap B) \subseteq \mathrm{Cl}_{(X,d)}(A) \cap \mathrm{Cl}_{(X,d)}(B).$$

You are asked to show that the set inclusion in part (b) can be strict in Exercise 3.6.

3.3 An alternative way of characterising closure

In this subsection we look at an alternative way of characterising the closure of a set – one that is more 'hands-on', in that it involves identifying points in the space that have a particular property. This alternative characterisation gives us another way of understanding the closure of a set.

Definition 3.6 *Closure point*

Let A be a subset of the metric space (X, d).

A point $x \in X$ is a **d-closure point** of A in X if for each $r > 0$, $B_d(x, r)$ meets the set A; that is, $B_d(x, r) \cap A \neq \emptyset$.

The idea is that a closure point is a point that (from the point of view of the metric) is 'very close' to A in that if x is a closure point, then there is a sequence of points in A that converges to it and so for each $r > 0$, the set A meets $B_d(x, r)$: $A \cap B_d(x, r) \neq \emptyset$. In particular, if $a \in A$, then a is certainly a closure point of A (regardless of the choice of metric), as shown in Figure 3.2.

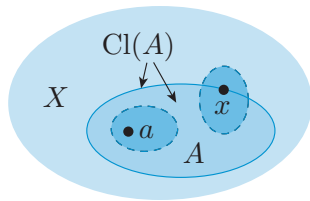


Figure 3.2 Closure points of A

Worked Exercise 3.7

Let (X, d) be a metric space and $A \subseteq X$. Show that $x \in X$ is a d-closure point of A in X if, and only if, there is a sequence of points from A, (a_n), say, that is d-convergent to x.

Solution

We need to show that:

1. If there is a sequence of points from A, (a_n), that is d-convergent to $x \in X$, then x is a d-closure point of A in X.
2. If $x \in X$ is a d-closure point of A in X, then there is a sequence of points from A, (a_n), say, that is d-convergent to x.

So, first, suppose that there is a sequence of points from A, (a_n), that is d-convergent to $x \in X$ and let $r > 0$ be given – we must show that $B_d(x, r) \cap A \neq \emptyset$. However, we know that $(d(a_n, x))$ is a real null sequence and so, in particular, there is $N \in \mathbb{N}$ for which $d(a_n, x) < r$ whenever $n > N$. Hence $a_n \in B_d(x, r)$ for $n > N$ and so $B_d(x, r) \cap A \neq \emptyset$, as required.

Conversely, suppose that $x \in X$ is a d-closure point of A in X; then for each $r > 0$, $B_d(x, r) \cap A \neq \emptyset$. In particular, for each $n \in \mathbb{N}$, we can find $a_n \in B_d(x, \frac{1}{n}) \cap A$. Consider the sequence (a_n): for each $n \in \mathbb{N}$, $a_n \in A$ and $d(a_n, x) < \frac{1}{n}$ and so $(d(a_n, x))$ is a real null sequence. Hence (a_n) is a sequence in A that is d-convergent to x, as required.

The next result tells us that the set of closure points of a given set is the same as the closure of that set, which justifies our choice of terminology.

Theorem 3.8

Let (X, d) be a metric space and let $A \subseteq X$. Then

$$\mathrm{Cl}_{(X,d)}(A) = \{x \in X : x \text{ is a } d\text{-closure point of } A \text{ in } X\}.$$

Proof Let $F = \{x \in X : x \text{ is a closure point of } A\}$ – we must show that $F = \mathrm{Cl}(A)$.

The proof proceeds in three steps:

1. Show that $A \subseteq F$.
2. Show that F is closed and so $\mathrm{Cl}(A) \subseteq F$.
3. Show that $F \subseteq \mathrm{Cl}(A)$.

Hence we start by recalling that each point $a \in A$ is a closure point of A and so certainly $A \subseteq F$.

To show that F is closed: suppose that (x_n) is a d-convergent sequence of points in F that converges to $x \in X$, say. We show that $x \in F$.

Since each point $x_n \in F$ is a closure point of A, then for each $n \in \mathbb{N}$, there is $a_n \in A$ for which $d(a_n, x_n) < \frac{1}{n}$. Hence by the Triangle Inequality, for each $n \in \mathbb{N}$,

$$d(a_n, x) \leq d(a_n, x_n) + d(x_n, x) < \tfrac{1}{n} + d(x_n, x).$$

But $(\frac{1}{n})$ and $(d(x_n, x))$ are both real null sequences and so $(d(a_n, x))$ is a real null sequence. In particular, for each $r > 0$, there is $n \in \mathbb{N}$ for which $d(a_n, x) < r$ and so $a_n \in B_d(x, r) \cap A \neq \emptyset$. Thus x is a closure point of A, that is, $x \in F$ and F is closed as claimed.

It only remains to show that $F \subseteq \mathrm{Cl}(A)$. To do this it is enough to show that for each $x \in F$, if E is a closed set that contains A, then $x \in E$. But this is straightforward: since x is a closure point of A, we can find a sequence (a_n) of elements of A that converges to x. But $A \subseteq E$ and so (a_n) is a sequence of elements of E that converges to x. Hence, since E is closed, $x \in E$ as required. ■

Worked Exercise 3.9

Show that for the Euclidean metric on $\mathbb{R}$, $\mathrm{Cl}(\mathbb{Q}) = \mathrm{Cl}(\mathbb{Q}^c) = \mathbb{R}$.

Solution

We begin by showing that $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$. We know that $\mathbb{Q} \subseteq \mathrm{Cl}(\mathbb{Q})$. So, since $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$, we need show only that the irrationals are closure points of $\mathbb{Q}$ to conclude, via Theorem 3.8, that $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$.

Let $x \in \mathbb{Q}^c$ and suppose that $r > 0$; then $B(x, r) = (x - r, x + r)$. By the density property of the reals, between any two distinct real numbers there is a rational number and so $(x - r, x + r)$ meets $\mathbb{Q}$. Thus $(x - r, x + r) \cap \mathbb{Q} \neq \emptyset$ and so x is a closure point of $\mathbb{Q}$. Hence $\mathbb{Q}^c \subseteq \mathrm{Cl}(\mathbb{Q})$, as required.

See the *Handbook*.

Similar arguments can be used to show that $\mathrm{Cl}(\mathbb{Q}^c) = \mathbb{R}$.

Exercise 3.6

By considering $\mathbb{Q}$ and $\mathbb{Q}^c$ for the Euclidean metric on $\mathbb{R}$, show that $\mathrm{Cl}(A \cap B)$ may be a *proper* subset of $\mathrm{Cl}(A) \cap \mathrm{Cl}(B)$.

Worked Exercise 3.10

Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Find $\mathrm{Cl}(A)$ for the Euclidean metric on $\mathbb{R}$.

Solution

Since $A \subseteq \mathrm{Cl}(A)$, it remains only to decide which points of A^c belong to $\mathrm{Cl}(A)$. Now $A \subseteq [0, 1]$, a closed set, and so, by Theorems 3.2 and 3.3,

$$\mathrm{Cl}(A) \subseteq \mathrm{Cl}([0, 1]) = [0, 1].$$

Hence we need only decide which points in $A^c \cap [0,1]$ are closure points of A.

Now $A^c \cap [0,1] = \{0\} \cup \bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n})$ and so if $x \in A^c \cap [0,1]$, then either $x = 0$ or there is $n \in \mathbb{N}$ for which $\frac{1}{n+1} < x < \frac{1}{n}$.

Suppose that $x \in A^c \cap [0,1]$.

If $x = 0$, then for each $r > 0$ there is $n \in \mathbb{N}$ for which $\frac{1}{n} < r$, and so $B(0,r) \cap A = (-r, r) \cap A \neq \emptyset$. Hence any ball centred on 0 intersects A. Thus 0 is a closure point of A.

If $x \in (\frac{1}{n+1}, \frac{1}{n})$ for some $n \in \mathbb{N}$, then for $0 < r \leq \min\{x - \frac{1}{n+1}, \frac{1}{n} - x\}$, we have $(x-r, x+r) \cap A = \emptyset$ (Figure 3.3). Hence, $B(x,r) = (x-r, x+r)$ does not intersect A for such an r, and so x is not a closure point of A.

Therefore $\mathrm{Cl}(A) = A \cup \{0\}$.

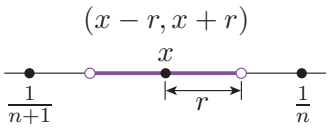


Figure 3.3 $x \in (\frac{1}{n+1}, \frac{1}{n})$

In Worked Exercise 1.11 we showed that for the discrete metric d_0, every subset of a space X is d_0-closed. In particular,

$$\mathrm{Cl}_{(\mathbb{R},d_0)}(\mathbb{Q}) = \mathbb{Q} \neq \mathbb{R} = \mathrm{Cl}_{(\mathbb{R},d^{(1)})}(\mathbb{Q})$$

and so different metrics can give rise to different closures.

Exercise 3.7

Let (a_n) be a strictly increasing sequence of real numbers that is bounded above. If $A = \{a_n : n \in \mathbb{N}\}$, find $\mathrm{Cl}(A)$ for the Euclidean metric on $\mathbb{R}$.

Recall that (a_n) is strictly increasing if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.

Hint: recall that an increasing sequence of real numbers that is bounded above converges to a limit $a \in \mathbb{R}$, with $a \geq a_n$ for each n.

Exercise 3.8

Let $X = \{(x_1, x_2) : x_1, x_2 \in \mathbb{Q}\}$. Show that for the usual Euclidean metric on the plane, $\mathrm{Cl}(X) = \mathbb{R}^2$.

We can relate closure to sequences.

Corollary 3.11

Let (X,d) be a metric space and let A be a subset of X. A point a belongs to the closure $\mathrm{Cl}_{(X,d)}(A)$ of A if, and only if, there is a sequence (a_n) in A that converges to a.

Proof Theorem 3.8 tells us that the closure of A coincides with the set of closure points of A, and the result of Worked Exercise 3.7 tells us that a point a is a closure point if, and only if, there is a sequence (a_n) in A that converges to a. ■

3.4 Closed sets and subspaces

The ability to associate a closed set with an arbitrary set by taking the closure gives us a way of identifying the closed subsets of a metric subspace (A, d_A) in terms of the closed sets of the original metric space (X, d).

Recall that d_A denotes the subspace (or induced) metric for A.

Theorem 3.12

Let (X, d) be a metric space and let $A \subseteq X$.

A set $E \subseteq A$ is d_A-closed if, and only if, there is a set F in X that is d-closed and for which

$$E = A \cap F.$$

In other words, closed sets in subspaces are just the intersections of the subspace with closed sets from the original space.

Proof Let F be d-closed in X. We show that $A \cap F$ is d_A-closed – that is, the limit of each d_A-convergent sequence in $A \cap F$ is in $A \cap F$.

Suppose that (a_n) is a d_A-convergent sequence in A for which $a_n \in A \cap F$ for each n. Then the limit of the sequence (for d_A), a, say, is in A and $(d_A(a_n, a))$ is a real null sequence. But $d_A(a_n, a) = d(a_n, a)$ for each $n \in \mathbb{N}$ and so a_n d-converges to a in X. But F is closed and so $a \in F$. Hence $a \in A \cap F$ and we conclude that $A \cap F$ is d_A-closed.

Now suppose that E is d_A-closed: we show that there is F in X that is d-closed for which $E = A \cap F$. Let $F = \mathrm{Cl}_{(X,d)}(E)$; this is certainly a d-closed set and we claim that $E = A \cap F$. We know that $E \subseteq A \cap \mathrm{Cl}_{(X,d)}(E)$ and so if we show that $A \cap \mathrm{Cl}_{(X,d)}(E) \subseteq E$, then we have finished.

So suppose that we can find $a \in (A \cap \mathrm{Cl}_{(X,d)}(E)) - E$; that is, suppose that $A \cap \mathrm{Cl}_{(X,d)}(E) \not\subseteq E$. Then $a \in A - E$ and $a \in \mathrm{Cl}_{(X,d)}(E)$. In particular, by Corollary 3.11 we can find a sequence (a_n) in E that d-converges to a. But, since $E \subseteq A$, it follows that each point $a_n \in A$ and so $d_A(a_n, a)$ is defined and equals $d(a_n, a)$, implying that $(d_A(a_n, a))$ is a real null sequence. Hence (a_n) is a sequence that d_A-converges to a. Since E is d_A-closed, this means that $a \in E$, which is a contradiction. We conclude that $A \cap \mathrm{Cl}_{(X,d)}(E) \subseteq E$. ■

4 Interior

In this section we return to the problem of associating an appropriate open set with a given subset of a metric space.

Recall that in Exercise 3.1 we looked at ways of associating an open set with various given subsets of $\mathbb{R}$ for the Euclidean metric, and the conclusions were as follows:

1. with $[0,1)$, $[0,1]$ and $(0,1)$, we associated the open set $(0,1)$
2. with $\{0\}$, we associated the (open) set $\emptyset$.

In all four cases, we found that the natural associated open set is the 'largest' open set that lies inside the starting set. This is not a coincidence. Suppose that we have been given a set A in a metric space (X,d) and we wish to find an open set inside it. There is always at least one such open set, namely the empty set, $\emptyset$. Now suppose that we have a family of open sets $\{U_i : i \in I\}$ each of which is contained in A; then

1. $\bigcup_{i\in I} U_i \subseteq A$, since $U_i \subseteq A$ for each $i \in I$
2. Theorem 2.15 tells us that $\bigcup_{i\in I} U_i$ is also an open set.

So, given any collection of open sets contained in A, their union is also an open set contained in A.

Consider now the collection

$$\{U : U \subseteq A \text{ and } U \text{ is } d\text{-open in } X\}.$$

Then, in view of the preceding, the set V that is defined by

$$V = \bigcup\{U : U \subseteq A \text{ and } U \text{ is } d\text{-open in } X\}$$

is an open set that is contained in A. Moreover if W is an open set that is contained in A, then W occurs in the union defining V and so $W \subseteq V \subseteq A$. In other words, V is the largest open set that is contained in A. This is our sought-after definition of what is known as the *interior* of a set A.

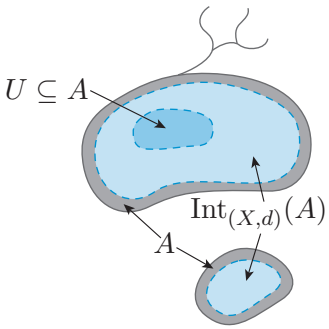


Figure 4.1 The interior of A

Definition 4.1 *Interior of a set*

Let (X,d) be a metric space and suppose $A \subseteq X$. We define the **interior of A in (X,d)**, $\mathrm{Int}_{(X,d)}(A)$, to be the largest d-open set in X that is contained in A (Figure 4.1). That is,

$$\mathrm{Int}_{(X,d)}(A) = \bigcup\{U : U \subseteq A \text{ and } U \text{ is } d\text{-open in } X\}.$$

Remarks

1. When it is clear from the context, we usually just write $\mathrm{Int}(A)$ and suppress mention of the metric space (X,d).
2. The interior of a set is always an open set.
3. It is always the case that $\mathrm{Int}(A) \subseteq A$.

Worked Exercise 4.2

Find the interior of $(0,1) \cup \{2\}$ for the Euclidean metric on $\mathbb{R}$.

Solution

Let $A = (0,1) \cup \{2\}$.

Since $\text{Int}(A) \subseteq A$ and $(0,1)$ is an open set contained in A, we know that

$$(0,1) \subseteq \text{Int}(A) \subseteq (0,1) \cup \{2\}.$$

Thus we need only determine whether 2 is in the interior of A. However, since $\text{Int}(A)$ is open, Theorem 2.9 tells us that $x \in \text{Int}(A)$ if, and only if, there is $r > 0$ so that $B(x,r) \subseteq \text{Int}(A)$. But for each $r > 0$, $2 + \frac{1}{2}r \in B(2,r)$ and $2 + \frac{1}{2}r \notin A$ and so cannot be in $\text{Int}(A)$. Hence $2 \notin \text{Int}(A)$ and we conclude that

$$\text{Int}((0,1) \cup \{2\}) = (0,1).$$

Exercise 4.1

Let (X,d) be a metric space. Find the interior (in (X,d)) of $\emptyset$ and X.

Exercise 4.2

Let (X,d) be a metric space. Show that if A and B are subsets of X with $A \subseteq B$, then $\text{Int}(A) \subseteq \text{Int}(B)$.

Theorem 4.3

Let (X,d) be a metric space and let $A, B \subseteq X$. Then:

(a) $\text{Int}_{(X,d)}(A) \cup \text{Int}_{(X,d)}(B) \subseteq \text{Int}_{(X,d)}(A \cup B)$

(b) $\text{Int}_{(X,d)}(A) \cap \text{Int}_{(X,d)}(B) = \text{Int}_{(X,d)}(A \cap B)$.

Proof We use the result of Exercise 4.2.

(a) Since both $A \subseteq A \cup B$ and $B \subseteq A \cup B$, it follows from Exercise 4.2 that $\text{Int}(A) \subseteq \text{Int}(A \cup B)$ and $\text{Int}(B) \subseteq \text{Int}(A \cup B)$. Hence

$$\text{Int}(A) \cup \text{Int}(B) \subseteq \text{Int}(A \cup B).$$

(b) Similarly, since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we obtain

$$\text{Int}(A \cap B) \subseteq \text{Int}(A) \cap \text{Int}(B).$$

$\text{Int}(A) \cap \text{Int}(B)$ is the intersection of two open sets and hence is open.

Moreover, $\text{Int}(A) \subseteq A$ and $\text{Int}(B) \subseteq B$, so $\text{Int}(A) \cap \text{Int}(B) \subseteq A \cap B$. Since, by Definition 4.1, $\text{Int}(A \cap B)$ is the largest open subset of $A \cap B$, it follows that

$$\text{Int}(A) \cap \text{Int}(B) \subseteq \text{Int}(A \cap B).$$

Therefore

$$\text{Int}(A) \cap \text{Int}(B) = \text{Int}(A \cap B). \quad \blacksquare$$

Exercise 4.3

For $\mathbb{R}$ with the Euclidean metric, let $A = [0, 1]$ and $B = (1, 2]$. Show that $\text{Int}(A) \cup \text{Int}(B)$ is a *proper* subset of $\text{Int}(A \cup B)$.

The use of Theorem 2.9 in Worked Exercise 4.2 suggests that it is possible to characterise the interior of a set via a property of the points that lie in it (in a similar way to what we did with the closure) and this is indeed the case, as the following result shows.

Theorem 4.4

Let (X, d) be a metric space and let $A \subseteq X$.

The interior of A is given by

$$\text{Int}_{(X,d)}(A) = \{x \in X : \text{there is } r > 0 \text{ so that } B_d(x, r) \subseteq A\}.$$

Proof We first show that

$$V = \{x \in X : \text{there is } r > 0 \text{ so that } B_d(x, r) \subseteq A\}$$

is an open set contained within the interior of A.

For $x \in V$, let r_x be chosen so that $B_d(x, r_x) \subseteq A$; then V can be written as

$$V = \bigcup_{x \in V} B_d(x, r_x) \subseteq A.$$

In particular, V is the union of a collection of open sets, so it is itself open, and it is contained within A, hence it is a subset of $\text{Int}(A)$: $V \subseteq \text{Int}(A)$.

We now show that $\text{Int}(A) \subseteq V$: if $\text{Int}(A) - V \neq \emptyset$, then there is $x \in \text{Int}(A) - V$ and so, since $\text{Int}(A)$ is an open set, we can use Theorem 2.9 to find $r > 0$ for which $B_d(x, r) \subseteq \text{Int}(A)$. But $\text{Int}(A) \subseteq A$ and so $B_d(x, r) \subseteq A$, implying that $x \in V$, which is a contradiction. Hence $\text{Int}(A) \subseteq V$.

Combining these two set inclusions, we conclude that $\text{Int}(A) = V$, as required. $\blacksquare$

This characterisation is very useful when trying to find the interior of a given set and so we give a name to points that have the property described in the above theorem.

Definition 4.5 *Interior point*

Let (X, d) be a metric space and let $A \subseteq X$.

We say $x \in X$ is a **d-interior point of A** if there is an $r > 0$ so that $B_d(x, r) \subseteq A$.

Theorem 4.4 then tells us that

$$\text{Int}_{(X,d)}(A) = \{x \in X : x \text{ is a } d\text{-interior point of } A\}.$$

Exercise 4.4

Find the interior of $\mathbb{Q}$ for the usual Euclidean metric on $\mathbb{R}$.

The interior of a set A is open and, since open sets are the complements of closed sets, one would expect there is a characterisation of the interior of a set in terms of closed sets – and indeed there is. The characterisation is useful for converting results about closure to results about interior.

Theorem 4.6

Let (X, d) be a metric space and let $A \subseteq X$. Then

$$\text{Int}_{(X,d)}(A) = \left[\text{Cl}_{(X,d)}(A^c)\right]^c = X - \text{Cl}_{(X,d)}(X - A).$$

Proof We first show that $[\text{Cl}(A^c)]^c \subseteq \text{Int}(A)$.

Certainly, by the definition of closure, $A^c \subseteq \text{Cl}(A^c)$ and so it follows, on taking complements in X, that

$$[\text{Cl}(A^c)]^c \subseteq (A^c)^c = A.$$

But $[\text{Cl}(A^c)]^c$ is an open set, and so, by the definition of the interior, $[\text{Cl}(A^c)]^c \subseteq \text{Int}(A)$.

We now show that $\text{Int}(A) \subseteq [\text{Cl}(A^c)]^c$.

Since $\text{Int}(A) \subseteq A$, it follows that

$$A^c \subseteq [\text{Int}(A)]^c.$$

Moreover, since $\text{Int}(A)$ is open, $[\text{Int}(A)]^c$ is closed. Hence, we use Theorems 3.2 and 3.3 to find

$$\text{Cl}(A^c) \subseteq \text{Cl}([\text{Int}(A)]^c) = [\text{Int}(A)]^c.$$

So, on taking complements, we have

$$\text{Int}(A) \subseteq [\text{Cl}(A^c)]^c.$$

Combining these two set inclusions gives the required result. ■

Worked Exercise 4.7

Find the interior of $[0,1]$ for the Euclidean metric on $\mathbb{R}$.

Solution

By Theorem 4.6,

$$\text{Int}([0,1]) = [\text{Cl}([0,1]^c)]^c = [\text{Cl}((-\infty,0)\cup(1,\infty))]^c.$$

But $\text{Cl}((-\infty,0)\cup(1,\infty)) = (-\infty,0]\cup[1,\infty)$ and so

$$\text{Int}([0,1]) = ((-\infty,0]\cup[1,\infty))^c = (0,1).$$

Exercise 4.5

Let (X,d) be a metric space and $A \subseteq X$. Show that A is d-open if, and only if, $\text{Int}_{(X,d)}(A) = A$.

5 Boundaries of sets

In this section we introduce the notion of the boundary of a set.

We have already defined the interior and closure of a set A, and we have seen that $\text{Int}(A) \subseteq A \subseteq \text{Cl}(A)$. The points that lie in $\text{Cl}(A) - \text{Int}(A)$ are special. For example, in $\mathbb{R}$ with the Euclidean metric, the interior of $(0,1]$ is $(0,1)$ and the closure is $[0,1]$, and so $\text{Cl}((0,1]) - \text{Int}((0,1]) = \{0,1\}$. The points 0 and 1 are 'left over'; they seem to be acting as *boundary points*. This observation motivates the following definition.

Definition 5.1 *Boundary*

Let (X,d) be a metric space and let $A \subseteq X$.

The **d-boundary of A in X** is the set

$$\text{Bd}_{(X,d)}(A) = \text{Cl}_{(X,d)}(A) - \text{Int}_{(X,d)}(A).$$

Remarks

1. When it is clear from the context, we usually just write $\text{Bd}(A)$ and suppress mention of the metric space (X,d).

2. For any metric space (X, d), we can deduce directly from the definition of boundary that both $\mathrm{Bd}(\emptyset) = \emptyset$ and $\mathrm{Bd}(X) = \emptyset$. In particular, for $A \subseteq X$, $\mathrm{Bd}_{(X,d)}(A)$ need not be the same as $\mathrm{Bd}_{(A,d_A)}(A)$, which is always the empty set.

Our first result shows that the name 'boundary' is an appropriate one.

Theorem 5.2

Let (X, d) be a metric space and let $A \subseteq X$. Then

$$\mathrm{Bd}_{(X,d)}(A) = \mathrm{Cl}_{(X,d)}(A) \cap \mathrm{Cl}_{(X,d)}(A^c).$$

In particular, $\mathrm{Bd}_{(X,d)}(A)$ is a closed set.

Proof This result follows from Theorem 4.6 in which we showed that $\mathrm{Int}(A) = \mathrm{Cl}(A^c)^c$, that is, $\mathrm{Int}(A) = X - \mathrm{Cl}(A^c)$. Hence

$$\begin{aligned}\mathrm{Bd}(A) &= \mathrm{Cl}(A) - \mathrm{Int}(A), \text{ by definition}\\ &= \mathrm{Cl}(A) \cap (\mathrm{Int}(A))^c\\ &= \mathrm{Cl}(A) \cap ((\mathrm{Cl}(A^c))^c)^c\\ &= \mathrm{Cl}(A) \cap \mathrm{Cl}(A^c),\end{aligned}$$

as claimed. ■

Exercise 5.1

Let (X, d) be a metric space and let $A \subseteq X$. Prove that

$$\mathrm{Bd}_{(X,d)}(A) = \mathrm{Bd}_{(X,d)}(A^c).$$

As with interior and closure, we can characterise the boundary of a set in terms of points with particular properties. These points are known as boundary points.

Definition 5.3 *Boundary point*

Let (X, d) be a metric space and let $A \subseteq X$.

A point $x \in X$ is a **d-boundary point of A** if for each $r > 0$, $B_d(x, r)$ meets both A and A^c (Figure 5.1).

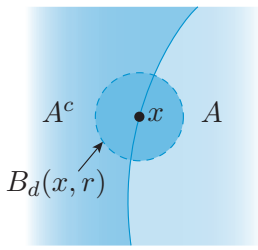


Figure 5.1 A boundary point of A

Theorem 5.4

Let (X, d) be a metric space and let $A \subseteq X$. Then

$$\mathrm{Bd}_{(X,d)}(A) = \{x \in X : x \text{ is a } d\text{-boundary point of } A\}.$$

Proof Let $B = \{x \in X : x \text{ is a } d\text{-boundary point of } A\}$ – we must show that $\mathrm{Bd}(A) = B$.

We show that a point $b \in B$ if, and only if, $b \in \mathrm{Bd}(A)$.

A point $b \in B$ if, and only if, for each $r > 0$, $B(b, r) \cap A \neq \emptyset$ and $B(b, r) \cap A^c \neq \emptyset$.

But $B(b, r) \cap A \neq \emptyset$ for each $r > 0$ if, and only if, b is a closure point of A, and $B(b, r) \cap A^c \neq \emptyset$ for each $r > 0$ if, and only if, b is a closure point of A^c.

Now b is a closure point of A if, and only if, $b \in \mathrm{Cl}(A)$, by Theorem 3.8, and b is a closure point of A^c if, and only if, $b \in \mathrm{Cl}(A^c)$, also by Theorem 3.8.

Hence $b \in B$ if, and only if, $b \in \mathrm{Cl}(A) \cap \mathrm{Cl}(A^c)$. But Theorem 5.2 tells us that $\mathrm{Cl}(A) \cap \mathrm{Cl}(A^c) = \mathrm{Bd}(A)$ and the result follows. ■

Theorem 5.5

Let (X, d) be a metric space and let $A \subseteq X$. Then

$$\mathrm{Bd}_{(X,d)}(A) = (\mathrm{Int}_{(X,d)}(A^c) \cup \mathrm{Int}_{(X,d)}(A))^c.$$

Proof We first show that $\mathrm{Bd}(A) \subseteq (\mathrm{Int}(A^c) \cup \mathrm{Int}(A))^c$ by using the characterisation of the boundary in terms of boundary points.

Let $x \in \mathrm{Bd}(A)$. Then x is not an interior point of A^c since $B_d(x, r) \cap A \neq \emptyset$ for each $r > 0$, and x is not an interior point of A since $B_d(x, r) \cap A^c \neq \emptyset$ for each $r > 0$. So

$$x \in \mathrm{Int}(A^c)^c \cap \mathrm{Int}(A)^c = (\mathrm{Int}(A^c) \cup \mathrm{Int}(A))^c,$$

by De Morgan's First Law. Thus

$$\mathrm{Bd}(A) \subseteq (\mathrm{Int}(A^c) \cup \mathrm{Int}(A))^c.$$

It remains only to verify that $(\mathrm{Int}(A^c) \cup \mathrm{Int}(A))^c \subseteq \mathrm{Bd}(A)$.

Let $x \in (\mathrm{Int}(A^c) \cup \mathrm{Int}(A))^c$. Then x is not an interior point of A, and so $B_d(x, r)$ intersects A^c for each $r > 0$. Similarly, x is not an interior point of A^c, and so $B_d(x, r)$ intersects A for each $r > 0$. Hence $B_d(x, r)$ intersects both A and A^c for each $r > 0$. So $x \in \mathrm{Bd}(A)$. Thus

$$(\mathrm{Int}(A^c) \cup \mathrm{Int}(A))^c \subseteq \mathrm{Bd}(A).$$

Therefore $\mathrm{Bd}(A) = (\mathrm{Int}(A^c) \cup \mathrm{Int}(A))^c$. ■

Remarks

1. Theorem 5.5 tells us immediately that $\mathrm{Bd}(A) \cup \mathrm{Int}(A^c) \cup \mathrm{Int}(A) = X$.
2. We deduce from Theorem 5.5 that $\mathrm{Bd}(A) \cap \mathrm{Int}(A^c) = \emptyset$ and $\mathrm{Bd}(A) \cap \mathrm{Int}(A) = \emptyset$. We saw earlier that $\mathrm{Int}(A) \cap \mathrm{Int}(A^c) = \emptyset$. Thus $\mathrm{Bd}(A)$, $\mathrm{Int}(A^c)$ and $\mathrm{Int}(A)$ are mutually disjoint.

The relationships between the interior and boundary of a set and its complement are illustrated in Figure 5.2. Note that $\mathrm{Bd}(A)$ is a closed set, disjoint from $\mathrm{Int}(A)$ and $\mathrm{Int}(A^c)$. It is difficult to do justice to this in a drawing.

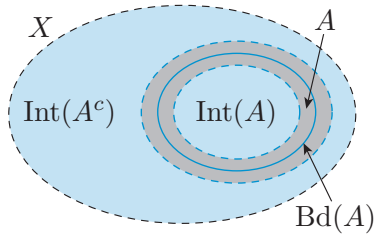


Figure 5.2 $\mathrm{Int}(A)$, $\mathrm{Int}(A^c)$ and $\mathrm{Bd}(A)$

Exercise 5.2

Let $\mathbb{R}$ have its Euclidean metric. Find the interior and boundary of (a) $\mathbb{Q}$ and (b) $\mathbb{Q}^c$.

Hint: use the result of Worked Exercise 3.9.

You are asked to consider the discrete metric in the next exercise.

Exercise 5.3

Let (X, d) be a metric space with the discrete metric and let A be a non-empty proper subset. Find the interior and boundary of A.

5.1 The algebra of boundaries

We turn now to some general results about the behaviour of interiors, exteriors and boundaries under unions and intersections.

Theorem 5.6

Let (X, d) be a metric space and let $A, B \subseteq X$. Then

$$\mathrm{Bd}_{(X,d)}(A \cup B) \subseteq \mathrm{Bd}_{(X,d)}(A) \cup \mathrm{Bd}_{(X,d)}(B).$$

Proof We use the characterisation of the boundary in terms of closure, given in Theorem 5.2, together with the behaviour of closures under unions and intersections given in Theorem 3.5:

We can also prove this result by using the results of Theorem 4.3.

$$\begin{aligned}
\mathrm{Bd}(A \cup B) &= \mathrm{Cl}(A \cup B) \cap \mathrm{Cl}((A \cup B)^c), && \text{by Theorem 5.2} \\
&= \mathrm{Cl}(A \cup B) \cap \mathrm{Cl}(A^c \cap B^c), && \text{by De Morgan's First Law} \\
&\subseteq (\mathrm{Cl}(A) \cup \mathrm{Cl}(B)) \cap (\mathrm{Cl}(A^c) \cap \mathrm{Cl}(B^c)), && \text{by Theorem 3.5} \\
&= (\mathrm{Cl}(A) \cap \mathrm{Cl}(A^c) \cap \mathrm{Cl}(B^c)) \cup (\mathrm{Cl}(B) \cap \mathrm{Cl}(A^c) \cap \mathrm{Cl}(B^c)) \\
&\subseteq (\mathrm{Cl}(A) \cap \mathrm{Cl}(A^c)) \cup (\mathrm{Cl}(B) \cap \mathrm{Cl}(B^c)) \\
&= \mathrm{Bd}(A) \cup \mathrm{Bd}(B), && \text{by Theorem 5.2.} \quad \blacksquare
\end{aligned}$$

Here we are using the identity $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$, which is a special case of the more general identity $(\cup_{i \in I} E_i) \cap G = \bigcup_{i \in I} (E_i \cap G)$ and is an example of a *distributive law*.

Exercise 5.4

Show that for all sets E, F, G the identity $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$ holds.

Exercise 5.5

Let (X, d) be a metric space and let $A, B \subseteq X$. Use Theorem 5.6 and the result of Exercise 5.1 to show that

$$\mathrm{Bd}_{(X,d)}(A \cap B) \subseteq \mathrm{Bd}_{(X,d)}(A) \cup \mathrm{Bd}_{(X,d)}(B).$$

6 The size of sets

In this section we look at two different ways of describing when sets are big or small. The first approach is metrical in nature, in that it measures the size of a set in terms of the metric of the underlying space. The second approach is a purely 'set theoretic' one that extends the familiar idea of saying that a finite set has n elements to the situation when the set is not finite. It does not depend on any notion of distance and has some surprising consequences: there are different 'sizes' of infinite sets.

6.1 Dense and nowhere dense sets

In this short subsection, we investigate two special properties that a subset of the underlying set X of a metric space may possess. These properties give an idea of the 'size' of a set but from a very particular viewpoint – that of the metric. The indicators of size that we will obtain are very crude and only allow for three possible values: 'small', 'big' and 'somewhere in between'. Despite this, the metrical approach has turned out to be useful in modern mathematics, mainly because the notion of 'big' that we will shortly define has some nice properties: the intersection of two 'big' open sets is itself a 'big' open set. (In fact it turns out that in many metric spaces the intersection of a countable collection of 'big' open sets is 'big' (but not necessarily open). This result, known as the Baire Category Theorem, has played a fundamental role in the development of modern mathematics, particularly in functional analysis.)

In Worked Exercise 3.9 we used the fact that there is a rational number between any two distinct real numbers – that is, that the rational numbers are *dense* in $\mathbb{R}$ – to help us show that if $\mathbb{R}$ carries its Euclidean metric, then $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$. In contrast, if $\mathbb{R}$ has its discrete metric, then the closure of every set is equal to the set itself. In particular, even an apparently very large set, such as $\mathbb{Q}$ or $\mathbb{R} - \mathbb{Q}$, would have closure not equal to $\mathbb{R}$. In fact, for the discrete metric, the only subset of $\mathbb{R}$ whose closure is $\mathbb{R}$ is $\mathbb{R}$ itself. A subset of $\mathbb{R}$ whose closure (with respect to a particular metric) is equal to the whole of $\mathbb{R}$ is called a *dense* subset of $\mathbb{R}$. More generally, we make the following definitions.

The closure of subsets for the discrete metric was discussed in Worked Exercise 1.11.

Definition 6.1 *Dense and nowhere dense*

Let (X, d) be a metric space and let $A \subseteq X$.

The set A is **d-dense in** X if $\mathrm{Cl}_{(X,d)}(A) = X$.

The set A is **nowhere dense in** X if $\mathrm{Int}_{(X,d)}(\mathrm{Cl}_{(X,d)}(A)) = \emptyset$.

Remarks

1. The notion of dense corresponds to 'big' and nowhere dense corresponds to 'small'.
2. Since $\mathrm{Cl}(X) = X$, the underlying set X of a metric space is always dense in itself.
3. For any metric space, since $\mathrm{Int}(\mathrm{Cl}(\emptyset)) = \mathrm{Int}(\emptyset) = \emptyset$, the empty set is nowhere dense.
4. A set A can be neither dense nor nowhere dense. Consider the interval $[0, 1]$ in $\mathbb{R}$ with the usual Euclidean metric: $\mathrm{Cl}([0, 1]) = [0, 1] \neq \mathbb{R}$ and

   $$\mathrm{Int}(\mathrm{Cl}([0, 1])) = \mathrm{Int}([0, 1]) = (0, 1) \neq \emptyset.$$

   Most sets are of this type – 'somewhere in between'.

Determining whether a given set is dense (or nowhere dense) in a space (X, d) is an exercise in the calculation of closures and interiors. In Worked Exercise 3.9 we saw that $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$ for the usual Euclidean metric and so $\mathbb{Q}$ is dense in $\mathbb{R}$ (for the Euclidean metric).

Worked Exercise 6.2

Let (X, d) be a metric space and let $A \subseteq X$ be dense in X. Show that if $A \subseteq B$ then B is also dense in X.

Solution

If $A \subseteq B$ then, by Theorem 3.2, $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$. But A is dense in X, so $\mathrm{Cl}(A) = X$. Hence $X = \mathrm{Cl}(A) \subseteq \mathrm{Cl}(B) \subseteq X$ and so $\mathrm{Cl}(B) = X$. That is, B is dense in X.

Exercise 6.1

Let (X, d) be a metric space and let $A \subseteq X$ be nowhere dense in X. Show that if $B \subseteq A$, then B is nowhere dense in X.

Exercise 6.2

Let (X, d) be a metric space and let U and V be d-open subsets of X that are d-dense in X.

Show that $U \cap V$ is also (a d-open subset of X that is) d-dense in X.

Dense sets and subspaces

If (A, d_A) is a metric subspace of the metric space (X, d) and we wish to find sets that are dense in A, then we need to take special care to track which metric we are calculating denseness with regard to. There is, however, a theorem that relates these two closures.

Theorem 6.3

Let (X, d) be a metric and let $A \subset X$. Let d_A denote the subspace metric for A.

A set B contained in A is d_A-dense in A if, and only if, $A \subseteq \mathrm{Cl}_{(X,d)}(B)$.

Proof This is a proof where it is important to track which metric we are using at each stage.

We need to show that $\mathrm{Cl}_{(A,d_A)}(B) = A$ if, and only if, $A \subseteq \mathrm{Cl}_{(X,d)}(B)$.

We show that $\mathrm{Cl}_{(X,d)}(B) \supseteq A$ implies $\mathrm{Cl}_{(A,d_A)}(B) = A$ by proving the contrapositive: $\mathrm{Cl}_{(A,d_A)}(B) \neq A$ implies $\mathrm{Cl}_{(X,d)}(B) \not\supseteq A$.

It is certainly the case that $\mathrm{Cl}_{(A,d_A)}(B) \subseteq A$, so suppose $a \in A - \mathrm{Cl}_{(A,d_A)}(B)$; then a is not a d_A-closure point of B and we can find $r > 0$ for which $B_{d_A}(a, r) \cap B = \emptyset$. That is,

$$\{x \in A : d_A(a, x) < r\} \cap B = \emptyset.$$

This set equality can be written as

$$B_d(a, r) \cap A \cap B = \emptyset.$$

But $B \subseteq A$ and so this is the same as

$$B_d(a, r) \cap B = \emptyset,$$

and so $a \notin \mathrm{Cl}_{(X,d)}(B)$. Hence $A \not\subseteq \mathrm{Cl}_{(X,d)}(B)$.

Now suppose that $\mathrm{Cl}_{(A,d_A)}(B) = A$ – we aim to show that $\mathrm{Cl}_{(X,d)}(B)$ contains A. So let $a \in A$ and $r > 0$. Since $\mathrm{Cl}_{(A,d_A)}(B) = A$, there is $b \in B$ with $d_A(a, b) < r$; but $d_A(a, b) = d(a, b)$ and so we deduce that $B_d(a, r)$ also meets B. Hence $a \in \mathrm{Cl}_{(X,d)}(B)$ and we conclude that $A \subseteq \mathbb{C}_{(X,d)}(B)$, as required. ■

This result lets us turn questions about dense sets in subspaces into questions about closures of sets in the original space.

6.2 Countable and uncountable sets

In Worked Exercise 3.9 we saw that $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$ for the usual Euclidean metric on $\mathbb{R}$ and so $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. This may not strike you as a particularly surprising result, but in this subsection we show that although $\mathbb{Q}$ is an infinite set, it is not as big as $\mathbb{R}$, having strictly fewer points. The results in this subsection may strike you as rather surprising and counterintuitive – after all, how can you have different sizes of infinity? But they are correct and their implications are profound.

We start by recalling that a *finite set* is one with finitely many elements, and that an *infinite set* is one that is not finite. In this subsection we restrict ourselves to categorising infinite sets into two types – those that are *countable* and those that are *uncountable*. Informally, we say that an infinite set is 'countable' if all its elements can be written down as an infinite 'list', one after another, so that they can then be *counted*. An infinite set that cannot be listed in this way is said to be 'uncountable'. (It is not immediately apparent that there are any sets that are uncountable and one of the main aims of this subsection is to show that some sets with which you are already familiar are, in fact, uncountable.)

The empty set, having no elements, is regarded as being a finite set.

In fact, using similar techniques to those described here, one can show that there are different types of uncountable set and that there is a whole hierarchy of different 'sizes' of infinity. We will not do that here.

For example, the elements of the set $\mathbb{N}$ of natural numbers can be listed in the following (obvious!) way:

$$1, 2, 3, \ldots;$$

so this set is countable (according to our informal definition). Similarly, the set $\mathbb{Z}$ of integers can be listed as

$$0, 1, -1, 2, -2, \ldots$$

Every integer appears somewhere in the list.

and so this set too is countable (according to our informal definition).

Note that if the elements of a set can be *listed*, then they can be *numbered*. For example, using our listing for the set $\mathbb{Z}$, we can say that 0 is the first term in the listing of $\mathbb{Z}$, 1 is the second term, -1 is the third term, and so on. Thus we can assign a natural number to each of them, as follows:

$$0 \mapsto 1,\ 1 \mapsto 2,\ -1 \mapsto 3,\ \ldots.$$

In effect, we have defined a one–one map from $\mathbb{Z}$ to $\mathbb{N}$.

However, it is not obvious whether we could list all the elements of the set $\mathbb{R}$ of real numbers.

These observations lead to the following definition.

Definition 6.4 *Countable and uncountable*

A set A is **countable** if there exists a one–one map $f\colon A \to \mathbb{N}$, and is **uncountable** otherwise.

A countable set with infinitely many elements is said to be **countably infinite**.

Some authors call a set A countable if there is a one–one *and onto* map from A to $\mathbb{N}$ – this alternative definition does *not* classify finite sets as countable.

Remarks

1. Since the definition does not require the one–one map to be *onto*, it follows that all finite sets (including $\emptyset$) are countable. For example, $\{7, \pi, \sqrt{2}, -31, \pi\sqrt{2}\}$ can be mapped one–one to $\{1, 2, 3, 4, 5\}$, a finite subset of $\mathbb{N}$, and so is countable.
2. If a set A is countable, so that there exists a one–one map from A to $\mathbb{N}$, then we can *list* its elements in the order defined by this one–one map. Conversely, if we can list the elements of a set A, then the ordering of the list defines a one–one map from A to $\mathbb{N}$. Hence a set is countable if, and only if, it is *listable*.
3. Every subset of a countable set is countable and this is shown in the worked exercise following these remarks. However, a subset of an uncountable set may be countable (finite or infinite) or uncountable.
4. The main goal of this subsection is to show you that there are some infinite sets that are uncountable. Until then, you will have to take it on trust that this is a sensible definition to make.

Worked Exercise 6.5

Suppose that $A \subseteq B$ and B is countable. Show that A is also countable.

Solution

Let $f\colon B \to \mathbb{N}$ be a one–one map. Such a map exists since we are assuming B is countable.

Define $f|_A\colon A \to \mathbb{N}$ by $f_A(a) = f(a)$ for each $a \in A$, so $f|_A$ is the restriction of f to the set A. Then $f|_A$ is also one–one, for if $f|_A(a) = f|_A(b)$, then $f(a) = f(b)$ and so $a = b$, since f is one–one. We conclude that A is countable.

We now investigate the countability of some of the standard subsets of $\mathbb{R}$. We begin by noting that the set $\mathbb{N}$ is clearly countable, since the map $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n$ is a one–one map from $\mathbb{N}$ to $\mathbb{N}$.

Exercise 6.3

Show that the set $\mathbb{Z}$ of integers is countable.

You may find the next result surprising.

Theorem 6.6

The set $\mathbb{Q}$ of rational numbers is countable.

Proof There are several ways of proving this result. Recall that any non-zero rational number can be written uniquely as $\frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ have no common factors. If we agree to give 0 the representation $\frac{0}{1}$, we have a unique representation of the form $\frac{p}{q}$ for each element of $\mathbb{Q}$. We now construct a one–one map from $\mathbb{Q}$ into $\mathbb{Z}$ as follows:

$$f : \frac{p}{q} \mapsto \begin{cases} 2^p 3^q, & \text{if } p \geq 0, \\ -2^{-p} 3^q, & \text{if } p < 0. \end{cases}$$

For example, $f(\frac{3}{4}) = 2^3\,3^4 = 648$ and $f(-\frac{3}{4}) = -2^{-(-3)}\,3^4 = -648$.

We know that $\mathbb{Z}$ is countable (by Exercise 6.3) and so there exists a one–one map from $\mathbb{Z}$ to $\mathbb{N}$. Composing these two maps, we obtain a one–one map from $\mathbb{Q}$ to $\mathbb{N}$. Thus $\mathbb{Q}$ is countable. ■

So far, we have looked only at countable subsets of $\mathbb{R}$, and you may be wondering whether *all* subsets of $\mathbb{R}$ are countable. In fact, there are infinitely many *un*countable subsets of $\mathbb{R}$. We now show that the subset of $\mathbb{R}$ comprising the open interval $(0, 1)$ is uncountable.

Theorem 6.7

The set $(0, 1)$ is uncountable.

Proof We prove this by supposing that $(0, 1)$ is countable and obtaining a contradiction.

If $(0, 1)$ is countable, then we can list all its elements in decimal form:

$$\begin{aligned} x_1 &= 0.\boxed{d_{1,1}}\, d_{1,2}\, d_{1,3} \ldots \\ x_2 &= 0.\, d_{2,1} \boxed{d_{2,2}}\, d_{2,3} \ldots \\ x_3 &= 0.\, d_{3,1}\, d_{3,2} \boxed{d_{3,3}} \ldots \\ &\vdots \quad \vdots \quad \vdots \end{aligned}$$

where each $d_{i,j}$ belongs to $\{0, 1, \ldots, 9\}$.

Terminating decimals such as 0.1 have two different decimal expansions: either 0.1000 or $0.0999\ldots$. In this case, we only use the 'terminating' expression that ends in an infinite number of 0s.

We now choose a real number

$$x = 0.a_1 a_2 a_3 \ldots$$

in such a way that

$$a_1 \neq d_{1,1},\ a_2 \neq d_{2,2},\ a_3 \neq d_{3,3},\ \ldots \text{ and } a_i \neq 9 \text{ for all } i.$$

By ensuring that $a_i \neq 9$ for all i, we have made certain that x does not end in an infinite sequence of 9s, and so is not equivalent to a number in our list ending in an infinite sequence of 0s.

For example, we could define a_n for $n \in \mathbb{N}$ by

$$a_n = \begin{cases} 0, & \text{if } d_{n,n} \neq 0, \\ 1, & \text{if } d_{n,n} = 0. \end{cases}$$

We have now obtained a real number x in $(0, 1)$ that does not appear in our list, since it differs from x_n at the nth decimal place. This, however, is a contradiction since the list was supposed to contain *all* the numbers in $(0, 1)$. Thus our original supposition that $(0, 1)$ is countable must be wrong, and so $(0, 1)$ is uncountable. ■

Corollary 6.8

The set $\mathbb{R}$ is uncountable.

Proof If $\mathbb{R}$ were countable, then the subset $(0, 1)$ would be countable. This contradicts Theorem 6.7, and hence $\mathbb{R}$ is uncountable. ■

Exercise 6.4

Show that any open interval (a, b) with $a < b$ is uncountable.

Hint: find a one–one map from $(0, 1)$ to (a, b).

By using the results of Worked Exercise 6.5 and Exercise 6.4, we can deduce that open intervals of the form (a, ∞) and $(-\infty, b)$ are uncountable, since each contains an uncountable subset (a, b) for $a < b$. (If (a, ∞) were countable, then Worked Exercise 6.5 implies that any subset of it would also be countable and we have just seen in Exercise 6.4 that (a, b) is uncountable whenever $a < b$.)

When dealing with metric spaces, we occasionally need to consider the union of two or more sets. In fact (perhaps surprisingly), the union of *countably* many countable sets is always countable.

Theorem 6.9

For each $n \in \mathbb{N}$, let A_n be a countable set. Then $A = \bigcup_{n=1}^{\infty} A_n$ is also countable.

Proof Since each set A_n is countable, we can list the elements of A_1 as

$$x_{1,1}, x_{1,2}, x_{1,3}, \ldots;$$

the elements of A_2 as

$$x_{2,1}, x_{2,2}, x_{2,3}, \ldots;$$

and so on. We can then list all the elements of A as shown below.

We start our list with the element $x_{1,1}$, at the top left of the array (whose subscripts sum to 2). Next, we take the elements $x_{1,2}$ and $x_{2,1}$ on the diagonal immediately to the right of $x_{1,1}$ (whose subscripts sum to 3), then the elements on the next diagonal (whose subscripts sum to 4), and so on.

$$\begin{array}{ccccccccccc}
x_{1,1} & \rightarrow & x_{1,2} & & x_{1,3} & \rightarrow & x_{1,4} & & x_{1,5} & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \\
x_{2,1} & & x_{2,2} & & x_{2,3} & & x_{2,4} & & \cdot & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & & \\
x_{3,1} & & x_{3,2} & & x_{3,3} & & x_{3,4} & & \cdot & \cdots \\
 & \swarrow & & \nearrow & & & & & & \\
x_{4,1} & & x_{4,2} & & x_{4,3} & & x_{4,4} & & \cdot & \cdots \\
\downarrow & \nearrow & & & & & & & & \\
x_{5,1} & & \cdot & & \cdot & & \cdot & & \cdot & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

This gives us a list starting $x_{1,1}, x_{1,2}, x_{2,1}, x_{3,1}, x_{2,2}, x_{1,3}, x_{1,4}, \ldots$.

Every element $x_{i,j}$ appears somewhere in the list.

Thus A is listable, and therefore countable. ■

There is an obvious corollary.

Corollary 6.10

The union of finitely many countable sets is countable.

Worked Exercise 6.11

Show that $\mathbb{Q} \times \mathbb{Q}$ is a countable subset of $\mathbb{R}^2$.

Solution

It suffices to observe that

$$\mathbb{Q} \times \mathbb{Q} = \bigcup_{p \in \mathbb{Q}} \{p\} \times \mathbb{Q}$$

and so $\mathbb{Q} \times \mathbb{Q}$ can be written as the countable union of countable sets. Hence Theorem 6.9 implies that $\mathbb{Q} \times \mathbb{Q}$ is countable.

7 Continuity and open and closed sets

In this short section we look briefly at how some of the ideas we have examined in this book may be extended further. In particular we look at the relationship between continuous functions and open and closed sets, and we obtain a characterisation of continuous functions between metric spaces in terms of both open and closed sets. This characterisation turns out to be the starting point for the theory of *point-set topology*; although this is outside the scope of this module, we give a brief overview of it.

This section is not assessed.

Theorem 7.1

Let (X, d) and (Y, e) be metric spaces and suppose that $f : X \to Y$. Then f is (d, e)-continuous on X if, and only if, for each e-closed set $E \subseteq Y$, $f^{-1}(E)$ is d-closed.

Proof *If:*

Suppose that f is (d, e)-continuous on X and let E be an e-closed subset of Y. We must show that $f^{-1}(E) = \{x \in X : f(x) \in E\}$ is a d-closed subset of X. So let (x_n) be a d-convergent sequence in X with limit x and suppose that $x_n \in f^{-1}(E)$ for each n. Our aim is to show that $x \in f^{-1}(E)$.

Then $(f(x_n))$ is a sequence in Y for which $f(x_n) \in E$ for each $n \in \mathbb{N}$. Moreover, since f is (d, e)-continuous, the sequence $(f(x_n))$ is (d, e)-convergent with limit $f(x)$. But E is e-closed and so $f(x) \in E$, implying that $x \in f^{-1}(E)$, as required.

Only if:

Suppose that if $E \subseteq Y$ is an e-closed set, then $f^{-1}(E)$ is d-closed. We must show that f is (d, e)-continuous; that is, for each d-convergent sequence (x_n) in X with limit x, say, we have $(f(x_n))$ is e-convergent with limit $f(x)$.

So suppose that (x_n) is a d-convergent sequence in X with limit x and yet $f(x_n)$ does not e-converge to $f(x)$. Then there is $\varepsilon > 0$ such that for infinitely many n, $f(x_n) \notin B_e(f(x), \varepsilon)$. By restricting attention to a subsequence of (x_n) if necessary, we may assume that for each n, $f(x_n) \notin B_e(f(x), \varepsilon)$. Now $B_e(f(x), \varepsilon)$ is an e-open set and so $Y - B_e(f(x), \varepsilon)$ is e-closed and $f(x_n) \in Y - B_e(f(x), \varepsilon)$ for each n. By hypothesis, $f^{-1}(Y - B_e(f(x), \varepsilon))$ is d-closed and we also have $x_n \in f^{-1}(Y - B_e(f(x), \varepsilon))$ by definition. We assumed that (x_n) was a convergent sequence and so the limit of (x_n), the point x, must lie in $f^{-1}(Y - B_e(f(x), \varepsilon))$, meaning that $f(x) \in Y - B_e(f(x), \varepsilon)$, which is impossible. So no such sequence (x_n) can exist and f is (d, e)-continuous as required. ■

Since open sets are the complement of closed sets in a metric space, we expect a version of this result in terms of open sets – and indeed, this is the case.

Corollary 7.2

Let (X, d) and (Y, e) be metric spaces and suppose that $f : X \to Y$. Then f is (d, e)-continuous on X if, and only if, for each e-open set $U \subseteq Y$, $f^{-1}(U)$ is d-open.

In order to prove this corollary, we need to know how complements of sets interact with map inverses.

Exercise 7.1

Let $f : X \to Y$ and $A \subseteq Y$. Show that

$$f^{-1}(Y - A) = X - f^{-1}(A).$$

We can now prove the corollary.

Proof of Corollary 7.2 Let U be an e-open set.

Then $Y - U$ is e-closed and so $f^{-1}(Y - U) = X - f^{-1}(U)$ is d-closed if, and only if, f is (d, e)-continuous.

But $X - f^{-1}(U)$ is d-closed if, and only if, $f^{-1}(U)$ is d-open.

Hence f is (d, e)-continuous if, and only if, for each e-open set U, $f^{-1}(U)$ is d-open. ■

Drawing together the results we have proved in this chapter, we make the following observations.

If (X, d) is a metric space, then the collection, $\mathcal{T}$ say, of *d-open sets* has the following properties:

(T1) Both $\emptyset$ and X are in $\mathcal{T}$.

(T2) If U and V are in $\mathcal{T}$, then so is $U \cap V$.

(T3) If $\{U_i : i \in I\}$ is a collection of sets that are all in $\mathcal{T}$, then

$$\bigcup_{i \in I} U_i \text{ is in } \mathcal{T}.$$

In addition, if $f\colon X \to Y$ and (Y, e) is a metric space, then f is (d, e)-continuous on X if, and only if, for each e-open set U in Y, $f^{-1}(U)$ is d-open in X.

These observations are the basis for a significant extension of the definition of continuity.

Definition 7.3 *Topology*

We define a **topology** for a set X to be any collection $\mathcal{T}$ of subsets of X that satisfy (T1), (T2) and (T3). We call the sets that are in $\mathcal{T}$ the *open sets*.

A *topological space* is a set X together with a collection of subsets of X, $\mathcal{T}$ say, that form a topology for X, denoted $(X, \mathcal{T})$.

Remarks

1. We have seen in this chapter how metric spaces give rise to a collection of sets that satisfy these conditions. A topology on a space X that is formed from a metric d in this way is called a *metrisable topology* and when we wish to emphasise the dependence on the underlying metric, we write $\mathcal{T}(d)$.

2. There are topologies that are not metrisable – that is, they are not the open sets that are associated with a metric. For a simple example of such a topology, consider $(\mathbb{R}, \mathcal{T})$ where $\mathcal{T} = \{\emptyset, \mathbb{R}\}$. It is easy to verify that $\mathcal{T}$ satisfies (T1)–(T3) and so defines a topology for $\mathbb{R}$. However, it is not one determined by a metric, since – by Lemma 1.2 – single points are closed sets for any metric d on $\mathbb{R}$; and so in particular the complement of a single point, such as $\mathbb{R} - \{0\}$, would be d-open. But $\mathbb{R} - \{0\}$ is not in $\mathcal{T}$ and so $\mathcal{T}$ cannot be generated from the metric – it is not a metrisable topology.

Moreover, in view of Corollary 7.2, given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ we can define what it means for a function to be continuous between them in a way that is consistent with the definition of continuity for metric spaces.

Definition 7.4 *Continuity in a topological space*

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f\colon X \to Y$. We say that f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous if for each $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$.

The main advantage of topologies is that there are spaces that are so 'large' it is not possible to define a metric for them that captures the properties that we may be interested in, and yet it is possible to define topologies for them that do.

Learning outcomes

We list those topics on which we may set assessment questions to test your understanding of the chapter.

After working through this chapter you should be able to:

(a) State and use the definition of a *closed set*.

(b) Determine whether a given set in a metric space is closed.

(c) State and use the definition of an *open set*.

(d) Determine whether a given set in a metric space is open.

(e) Describe and use the basic properties of open and closed sets.

(f) Use various results concerning the union and intersection of sets, including De Morgan's Laws.

(g) Find the closure points and closure of a given set.

(h) Find the interior points and interior of a given set.

(i) Find the boundary points and boundary of a given set.

(j) Describe and use the basic properties of closures, interiors and boundaries.

(k) Determine whether a given set is dense or nowhere dense.

(l) Distinguish between finite, countably infinite and uncountable sets.

Solutions to exercises

Solution to Exercise 1.1

(a) (i) The limit of the sequence, 0, is in A.

(ii) The limit of the sequence, 0, is not in A.

(b) (i) The limit of the sequence, $(1, 2)$, is in A.

(ii) The limit of the sequence, $(1, 2)$, is not in A.

Solution to Exercise 1.2

The terms of the sequence (x_n), where $x_n = a - \frac{1}{n}$, all lie in $(-\infty, a)$. However, the limit of the sequence (with respect to the Euclidean metric d) is a, which does not lie in $(-\infty, a)$. Hence $(-\infty, a)$ is not closed for the Euclidean metric.

Solution to Exercise 1.3

(a) $A^c = \{2\}$; (b) $A^c = \{2, 4, 5\}$.

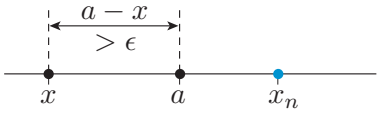
Solution to Exercise 1.4

$A^c = \mathbb{R} - [0, 1] = (-\infty, 0) \cup (1, \infty)$.

Solution to Exercise 1.5

The complement of $(-\infty, a)$ in $\mathbb{R}$ is $[a, \infty)$ and the argument to show that this set is d-closed mirrors that of Worked Exercise 1.4.

Let (x_n) be a d-convergent sequence in $\mathbb{R}$ with limit x, and suppose that $x_n \in [a, \infty)$ for each $n \in \mathbb{N}$. In order to show that the interval $[a, \infty)$ is d-closed we need to show that $x \in [a, \infty)$. Thus our goal is to show that $x \geq a$.



So suppose instead that $x < a$; then $|x - x_n| = x_n - x \geq a - x > 0$ for each $n \in \mathbb{N}$. Thus if we choose $\varepsilon > 0$ for which $a - x > \varepsilon$, as shown in the figure, then we have $|x - x_n| > \varepsilon$ for each $n \in \mathbb{N}$, contradicting the fact that $x_n \to x$ as $n \to \infty$. Hence $x \geq a$ and $[a, \infty)$ is d-closed as required.

Solution to Exercise 1.6

Now

$$\begin{aligned} A &= \{y \in X : d(x, y) \geq r\} \\ &= X - \{y \in X : d(x, y) < r\} \\ &= X - B_d(x, r). \end{aligned}$$

Hence, by the result of Worked Exercise 1.6, A is d-closed.

Solution to Exercise 1.7

Let $a = \lim_{n\to\infty} a_n$ and suppose that $a < m$. Then there is $\varepsilon > 0$ such that $a < m - \varepsilon$. Hence, we can calculate that for each $n \in \mathbb{N}$,

$$a < m - \varepsilon \leq a_n - \varepsilon$$

and so rearranging gives

$$\varepsilon < a_n - a \text{ for each } n \in \mathbb{N}.$$

This means that $(|a_n - a|)$ is not a null sequence and so $a_n \not\to a$ as $n \to \infty$, which is a contradiction. Thus $a \geq m$.

Similarly, if $M < a$, then there is $\varepsilon > 0$ such that $M + \varepsilon < a$. Hence, we can calculate that for each $n \in \mathbb{N}$,

$$a_n \leq M < M + \varepsilon < a$$

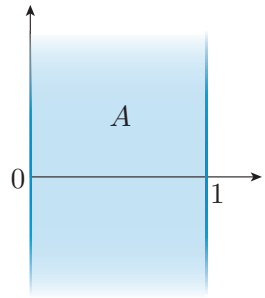
and rearranging gives

$$\varepsilon < a - a_n \text{ for each } n \in \mathbb{N}.$$

This means that $(|a_n - a|)$ is not a null sequence and so $a_n \not\to a$ as $n \to \infty$, which is a contradiction. Thus $a \leq M$.

Solution to Exercise 1.8

(a) A sketch is shown here. Note that the convention for boundaries is to draw them with solid lines if the boundary is included in the set and with dashed lines if it is not.



(b) Let $(\mathbf{a}_n = (a_{1,n}, a_{2,n}))$ be a convergent sequence in the plane, all of whose terms lie in A. So $(\mathbf{a}_n)$ has limit $\mathbf{a} = (a_1, a_2)$, say, and for each $n \in \mathbb{N}$, $0 \leq a_{1,n} \leq 1$.

Since $(\mathbf{a}_n)$ is convergent, we know from Theorem 3.15 of Chapter 15 that both of the real sequences $(a_{1,n})$ and $(a_{2,n})$ are also convergent with limits a_1 and a_2.

We now immediately use Lemma 1.8 to deduce that $0 \leq a_1 \leq 1$, implying that $\mathbf{a} \in A$. Hence A is closed for the Euclidean metric.

Solution to Exercise 1.9

Suppose that (f_n) is a $d_{\max}$-convergent sequence of functions in $C[0,1]$ with $f_n \in B_M$ for each n. We must show that the limit of the sequence, f, say, also lies in B_M. That is, we must show $f(0) = M$.

For each $n \in \mathbb{N}$ we have

$$|f(0) - M| = |f(0) - f_n(0)|$$

because $f_n \in B_M$. Hence

$$|f(0) - M| \leq d_{\max}(f_n, f),$$

which is a null sequence. We conclude that

$$|f(0) - M| \leq 0,$$

implying that $f(0) = M$ and so $f \in B_M$. Hence B_M is $d_{\max}$-closed.

Solution to Exercise 1.10

Suppose that (f_n) is a $d_{\max}$-convergent sequence of functions in $C[0,1]$ with $f_n \in C$ for each n. We must show that the limit of the sequence, f, say, also lies in C. That is, we must show $|f(a)| \leq M$.

For each $n \in \mathbb{N}$, we can use the Triangle Inequality to find an upper bound for $|f(a)|$ involving f_n as follows:

$$\begin{aligned} |f(a)| = |(f(a) - f_n(a)) + f_n(a)| &\leq |f(a) - f_n(a)| + |f_n(a)| \\ &\leq |f(a) - f_n(a)| + M, \end{aligned}$$

since $f_n \in C$. But $|f(a) - f_n(a)| \leq d_{\max}(f_n, f)$ and so

$$|f(a)| \leq d_{\max}(f_n, f) + M \text{ for each } n \in \mathbb{N}.$$

Yet $(d_{\max}(f_n, f))$ is a null sequence and so we conclude that

$$|f(a)| \leq M,$$

implying that $f \in C$. Thus C is $d_{\max}$-closed, as required.

Solution to Exercise 1.11

We saw in Lemma 1.2 that the singleton sets $\{x_i\}$ are d-closed for each $i = 1, 2, \ldots, n$ and so we can use the remark following Proposition 1.14 to conclude that their union,

$$\{x_1, x_2, \ldots, x_n\} = \bigcup_{i=1}^{n} \{x_i\},$$

is also d-closed.

Solution to Exercise 1.12

There are many possible solutions to this question and your answer may differ from ours.

Let $A_n = \{\frac{1}{n}\}$ for $n \in \mathbb{N}$. Then each set A_n is a singleton and so is closed by Lemma 1.2. However, by considering the sequence (a_n) given by $a_n = \frac{1}{n}$, which has limit 0, we see that $A = \cup_{n=1}^{\infty} A_n = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not a closed set, since (a_n) is an example of a convergent sequence with terms lying in A whose limit is not in A.

Solution to Exercise 1.13

(a) $K_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and $K_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.

(b) Each K_n is a finite union of closed intervals (each of which is closed with respect to the Euclidean metric) and so, by Proposition 1.14, K_n is itself closed for the Euclidean metric.

Solution to Exercise 1.14

We observe that the sphere (in X) can be written as the intersection of two closed sets:

$$\begin{aligned} S_d(a,r) &= \{x \in X : d(a,x) \leq r\} \cap \{x \in X : d(a,x) \geq r\} \\ &= B_d[a,r] \cap \{x : d(a,x) \geq r\} \\ &= B_d[a,r] \cap (B_d(a,r))^c, \end{aligned}$$

and so, by Proposition 1.17, $S_d(a,r)$ is closed.

We showed in Worked Exercise 1.6 that $B_d(a,r)^c$ is d-closed.

Solution to Exercise 1.15

(a) The argument is similar to that given in the solution to Exercise 1.9.

Suppose that (f_n) is a $d_{\max}$-convergent sequence of functions in $C[0,1]$ with $f_n \in F_M$ for each n. We must show that the limit of the sequence – f, say – also lies in F_M. That is, we must show $f(\frac{1}{M}) = 0$.

For each $n \in \mathbb{N}$ we can find an upper bound for $|f(\frac{1}{M})|$ involving f_n as follows:

$$\begin{aligned} \left|f(\tfrac{1}{M})\right| &= \left|f(\tfrac{1}{M}) - 0\right| \\ &= \left|f(\tfrac{1}{M}) - f_n(\tfrac{1}{M})\right| \\ &\leq d_{\max}(f, f_n). \end{aligned}$$

But $(d_{\max}(f_n, f))$ is a null sequence and so we conclude (by the Squeeze Rule) that

$$|f(\tfrac{1}{M})| = 0,$$

implying that $f \in F_M$. We conclude that F_M is $d_{\max}$-closed.

(b) The function $f\colon [0,1] \to \mathbb{R}$ given by $f(x) = 0$ for each $x \in [0,1]$ lies in the intersection. In fact, any function $g \in C[0,1]$ that is zero at each of the points $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ will lie in the intersection of the sets. (Notice that, because functions in $C[0,1]$ are continuous, if a function $g \in C[0,1]$ is zero at each of $1, \frac{1}{2}, \frac{1}{3}, \ldots$, then it must also be zero at zero: $g(0) = 0$.)

(c) Since, by part (a), each of the sets F_M is $d_{\max}$-closed, we conclude from Proposition 1.17 that their intersection is also $d_{\max}$-closed.

Solution to Exercise 2.1

To show that (a,∞) is open, we show that its complement in $\mathbb{R}$ is closed: $\mathbb{R} - (a,\infty) = (-\infty, a]$ and we saw in Worked Exercise 1.4 that this set is closed for the Euclidean metric.

The interval $[a,\infty)$ is not open for the Euclidean metric because the complement of $[a,\infty)$ (in $\mathbb{R}$) is $(-\infty, a)$ and this is not a closed set: the sequence (a_n) given by $a_n = a - \frac{1}{n}$ lies inside $(-\infty, a)$ for each n but the limit does not.

Solution to Exercise 2.2

The complement of the interval $(a, b]$ in $\mathbb{R}$ is $(-\infty, a] \cup (b, \infty)$ and by considering the sequence $x_n = b + \frac{1}{n}$ for $n \in \mathbb{N}$, we see that the complement is not closed for the Euclidean metric.

Similarly, we see that the complement of the interval $[a, b]$ in $\mathbb{R}$ is $(-\infty, a) \cup (b, \infty)$ and consideration of the same sequence shows that this is not a closed set for the Euclidean metric.

Solution to Exercise 2.3

If we look at the complement of C_M in $C[0, 1]$, we see that

$$C[0, 1] - C_M = \{f \in C[0, 1] : |f(x)| \leq M \text{ for each } x \in [0, 1]\}$$

and this is the set that we determined to be closed (for $d_{\max}$) in Worked Exercise 1.12. Hence C_M is open.

Solution to Exercise 2.4

Let $f_0 \in C[0, 1]$ be the function defined by

$$f_0(x) = 0 \quad \text{for each } x \in [0, 1].$$

Then $f_0 \in S$. We show that there is no $r > 0$ so that $B_{d_{\max}}(f_0, r)$ lies inside S.

So let $r > 0$. We must find a function g with $d_{\max}(f_0, g) < r$, but $g(0) \neq 0$ and so $g \notin S$.

Let $g \in C[0, 1]$ be the function given by

$$g(x) = \tfrac{1}{2}r \quad \text{for each } x \in [0, 1].$$

Then $d_{\max}(f_0, g) = \frac{1}{2}r < r$, and so $g \in B_{d_{\max}}(f_0, r)$. But $g(0) = \frac{1}{2}r \neq 0$ and so $g \notin S$. Thus, S is not open.

Solution to Exercise 2.5

Combining the results of Worked Exercise 2.4 and Exercise 2.3 of Chapter 15, we see that for $\mathbf{a} \in \mathbf{C}$ and $n \in \mathbb{N}$,

$$B_{d_{\mathbf{C}}}(\mathbf{a}, 2^{-n}) = B_{d_{\mathbf{C}}}[\mathbf{a}, 2^{-(n+1)}] = (a_1, a_2, \ldots, a_n, c_1, c_2, \ldots) : (c_i) \in \mathbf{C}.$$

Thus open balls and closed balls coincide in $\mathbf{C}$, and so it follows from Proposition 2.3 that the $d_{\mathbf{C}}$-closed balls are $d_{\mathbf{C}}$-open!

Solution to Exercise 2.6

Since (a_n) does not converge to a, there is $\varepsilon > 0$ such that for each $N \in \mathbb{N}$ there is an $n > N$ with $a_n \notin B_d(a, \varepsilon)$. Let (a_{m_n}) be the subsequence of (a_n) formed as follows:

- Let m_1 be the first integer such that $a_{m_1} \notin B_d(a, \varepsilon)$.
- Let $m_2 > m_1$ be the first integer strictly larger than m_1 such that $a_{m_2} \notin B_d(a, \varepsilon)$.
- For $n > 2$, let m_n be the first integer strictly larger than m_{n-1} for which $a_{m_n} \notin B_d(a, \varepsilon)$.

Then (a_{m_n}) is a subsequence with the required property.

Solution to Exercise 2.7

$A_1 = \{2, 4, 6, 8, 10, 12\}$, $A_2 = \{3, 6, 9, 12\}$;
$A_1 \cap A_2 = \{6, 12\}$, $A_1 \cup A_2 = \{2, 3, 4, 6, 8, 9, 10, 12\}$.
Thus, $A_1^c = \{1, 3, 5, 7, 9, 11\}$; $A_2^c = \{1, 2, 4, 5, 7, 8, 10, 11\}$;
$(A_1 \cup A_2)^c = \{1, 5, 7, 11\} = A_1^c \cap A_2^c$;
$(A_1 \cap A_2)^c = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11\} = A_1^c \cup A_2^c$.

Solution to Exercise 2.8

First Law: $(A_1 \cup A_2 \cup A_3)^c = A_1^c \cap A_2^c \cap A_3^c$. In other words,

$$X - (A_1 \cup A_2 \cup A_3) = (X - A_1) \cap (X - A_2) \cap (X - A_3).$$

Second Law: $(A_1 \cap A_2 \cap A_3)^c = A_1^c \cup A_2^c \cup A_3^c$. In other words,

$$X - (A_1 \cap A_2 \cap A_3) = (X - A_1) \cup (X - A_2) \cup (X - A_3).$$

Solution to Exercise 2.9

Let (X, d) be a metric space, and let U be the union of a collection of d-open sets $\{U_i : i \in I\}$.

We must show that U is d-open: that is, $X - U$ is d-closed. We use De Morgan's First Law to find

$$\begin{aligned} X - U &= X - \left(\bigcup_{i \in I} U_i \right) \\ &= \bigcap_{i \in I} (X - U_i). \end{aligned}$$

But for each $i \in I$, $X - U_i$ is d-closed since U_i is d-open and so, by Proposition 1.17, $\bigcap_{i \in I}(X - U_i)$ is d-closed. That is, $X - U$ is d-closed and so U is d-open, as required.

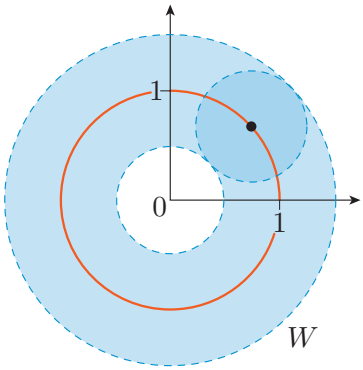
Solution to Exercise 2.10

(a) For each $(x, y) \in U$, $y - 1 > 0$. Let $r_{x,y} = y - 1$ and consider the open balls $B_d((x, y), r_{x,y})$. U is the union of these infinitely many open balls, that is,

$$U = \bigcup_{(x,y)\in U} B_d((x, y), r_{x,y})$$

and so U is the union of (an infinite collection of) open sets. Therefore, by Theorem 2.15, U is d-open.

(b) $V = B_d((1, 1), 1) \cap B_d((2, 2), 1)$; that is, V is the intersection of two open balls. Therefore, by Theorem 2.13, V is open.



(c) The point (x, y) belongs to W if, and only if, it is at a distance less than $\frac{1}{2}$ from some point (a, b) on the unit circle. That is, W is the union of an infinite collection of open balls of radius $\frac{1}{2}$, as shown in the figure:

$$W = \bigcup_{(a,b):a^2+b^2=1} B_d((a, b), \tfrac{1}{2}).$$

Hence by Theorem 2.15, W is open.

Solution to Exercise 3.1

Since we do not yet have any defined method for associating an open or closed set with a given set, you can view your answers as being correct provided that they are of the correct kind (open or closed), regardless of whether they match ours!

(a) Proceeding by analogy with the interval $(0, 1]$, we would associate the closed interval $[0, 1]$ and the open interval $(0, 1)$.

(b) The interval $[0, 1]$ is closed and so the natural closed set to associate with it is itself, $[0, 1]$.

For an open set: there are many open sets that contain the interval $[0, 1]$ – for example any interval of the form (a, b), where $a < 0$ and $b > 1$ – and there appears to be no canonical way to choose a particular one. However, if we look at open sets that are contained in $[0, 1]$, then we discover that there appears to be a 'largest' such set, namely the open interval $(0, 1)$ – any other open set inside $[0, 1]$ lies inside this open interval, and so $(0, 1)$ is a natural open set to associate with $[0, 1]$.

(c) The interval $(0, 1)$ is open and so the natural open set to associate with it is itself, $(0, 1)$.

For a closed set: there are many closed sets that lie inside the interval $(0, 1)$ – for example any interval of the form $[a, b]$, where $0 < a \leq b < 1$ – and there appears to be no canonical way to choose one. However, if we look at closed sets that contain $(0, 1)$, then we discover that there appears to be a 'smallest' such set, namely the closed interval $[0, 1]$, and so this is a natural closed set to associate with $[0, 1]$.

(d) This is a set consisting of a single point and so is a closed set itself. Thus the naturally associated closed set is $\{0\}$.

To associate an open set is a little trickier: there are lots of open sets that contain $\{0\}$ – for example, any interval of the form (a, b) where $a < 0$ and $b > 0$ – and none of them seem more natural than any other. Regarding open subsets of $\{0\}$, our choice here is limited: the only open subset of $\{0\}$ is the empty set, $\emptyset$, and we choose this as our natural open set associated with $\{0\}$.

Solution to Exercise 3.2

The empty set is a closed set (that contains itself) and so

$$\emptyset \subseteq \mathrm{Cl}_{(X,d)}(\emptyset) \subseteq \emptyset.$$

Hence $\mathrm{Cl}_{(X,d)}(\emptyset) = \emptyset$.

Similarly X is a closed set (that contains itself) and so

$$X \subseteq \mathrm{Cl}_{(X,d)}(X) \subseteq X.$$

Hence $\mathrm{Cl}_{(X,d)}(X) = X$.

Solution to Exercise 3.3

We know that $(0, 1)$ is not closed for the Euclidean metric and that $(0, 1) \subseteq \mathrm{Cl}((0, 1))$. We also know that $[0, 1]$ is a closed set that contains $(0, 1)$ and so

$$(0, 1) \subseteq \mathrm{Cl}((0, 1)) \subseteq [0, 1].$$

It remains only to show that there are no proper subsets of $[0, 1]$ that contain $(0, 1)$ and are closed. By considering the sequence $\left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}$ we see that $(0, 1]$ is not closed and a similar argument also shows that $[0, 1)$ is not closed. Thus there are no closed sets F lying between $(0, 1)$ and $[0, 1]$. Hence $\mathrm{Cl}((0, 1)) = [0, 1]$.

Solution to Exercise 3.4

Following the method of the solution to Exercise 1.15, we observe that

$$\begin{aligned} A &= \{f \in C[0, 1] : f(x) = 0 \text{ for } x \in I\} \\ &= \bigcap_{x\in I} \{f \in C[0, 1] : f(x) = 0\}. \end{aligned}$$

This is an intersection of closed sets. It follows from Proposition 1.17 that A is closed, and so, by Theorem 3.3, $\mathrm{Cl}(A) = A$.

Solution to Exercise 3.5

It is certainly the case that both $A \subseteq \mathrm{Cl}(A)$ and $B \subseteq \mathrm{Cl}(B)$ and so

$$A \cap B \subseteq \mathrm{Cl}(A) \cap \mathrm{Cl}(B),$$

but $\mathrm{Cl}(A) \cap \mathrm{Cl}(B)$ is a closed set and so

$$\mathrm{Cl}(A \cap B) \subseteq \mathrm{Cl}(A) \cap \mathrm{Cl}(B),$$

as required.

Solution to Exercise 3.6

By the definition of complements, $\mathbb{Q} \cap \mathbb{Q}^c = \emptyset$ and, by the definition of closure, $\mathrm{Cl}(\emptyset) = \emptyset$. Thus

$$\mathrm{Cl}(\mathbb{Q} \cap \mathbb{Q}^c) = \emptyset.$$

However, by Worked Exercise 3.9, $\mathrm{Cl}(\mathbb{Q}) = \mathrm{Cl}(\mathbb{Q}^c) = \mathbb{R}$, and so

$$\mathrm{Cl}(\mathbb{Q}) \cap \mathrm{Cl}(\mathbb{Q}^c) = \mathbb{R}.$$

Therefore, for $A = \mathbb{Q}$ and $B = \mathbb{Q}^c$, $\mathrm{Cl}(A \cap B)$ is a proper subset of $\mathrm{Cl}(A) \cap \mathrm{Cl}(B)$.

Solution to Exercise 3.7

Since (a_n) is an increasing sequence that is bounded above, it has a limit, a, say. We know that $A \subseteq \mathrm{Cl}(A)$, so it only remains to determine which points in A^c belong to $\mathrm{Cl}(A)$.

If $x \in A^c$ and $x \neq a$, then either $x < a_1$, or $x > a$, or (since (a_n) is strictly increasing) there is an $n \in \mathbb{N}$ for which

$$a_n < x < a_{n+1}.$$

In each case, we can find $r > 0$ such that $B(x, r) \cap A = (x - r, x + r) \cap A = \emptyset$, and so x is not a closure point of A.

If $x = a$, then for each $r > 0$, the definition of convergence of a sequence means that there is an $N \in \mathbb{N}$ such that $a_n \in B(a, r)$ for each $n > N$. Hence $B(a, r) \cap A = (a - r, a + r) \cap A \neq \emptyset$. Thus a is a closure point of A.

Therefore

$$\mathrm{Cl}(A) = A \cup \{a\}.$$

Solution to Exercise 3.8

Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and let $r > 0$ be given.

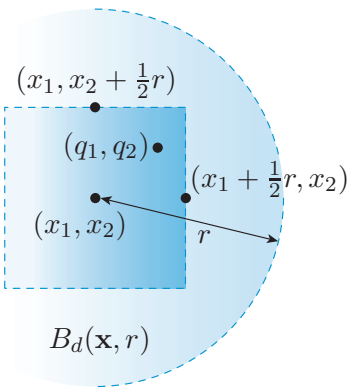
The density property of the reals means that between any two distinct real numbers there is a rational number, and so $(x_1 - \frac{1}{2}r, x_1 + \frac{1}{2}r) \cap \mathbb{Q} \neq \emptyset$ and $(x_2 - \frac{1}{2}r, x_2 + \frac{1}{2}r) \cap \mathbb{Q} \neq \emptyset$. Hence

$$((x_1 - \tfrac{1}{2}r, x_1 + \tfrac{1}{2}r) \times (x_2 - \tfrac{1}{2}r, x_2 + \tfrac{1}{2}r)) \cap (\mathbb{Q} \times \mathbb{Q}) \neq \emptyset.$$

Moreover,

$$(x_1 - \tfrac{1}{2}r, x_1 + \tfrac{1}{2}r) \times (x_2 - \tfrac{1}{2}r, x_2 + \tfrac{1}{2}r) \subseteq B_d(\mathbf{x}, r).$$

Hence $B_d(\mathbf{x}, r)$ intersects X, as shown in the figure.



Thus each open ball around $\mathbf{x}$ meets X implying, by Theorem 3.8, that $\mathbf{x} \in \mathrm{Cl}(X)$. Thus $\mathbb{R}^2 \subseteq \mathrm{Cl}(X) \subseteq \mathbb{R}^2$, as required.

Solution to Exercise 4.1

Since $\emptyset$ is an open set, $\emptyset \subseteq \text{Int}(\emptyset)$ but we also know that $\text{Int}(\emptyset) \subseteq \emptyset$ and so $\text{Int}(\emptyset) = \emptyset$.

Similarly, X is an open set and so

$$X \subseteq \text{Int}(X) \subseteq X,$$

giving $\text{Int}(X) = X$.

Solution to Exercise 4.2

Suppose that $A \subseteq B$; then $\text{Int}(A)$ is an open subset of X that is contained in A and so it is also an open set contained in B. Hence, by the definition of $\text{Int}(B)$, $\text{Int}(A) \subseteq \text{Int}(B)$.

Solution to Exercise 4.3

We have $A \cup B = [0, 2]$ and so $\text{Int}(A \cup B) = (0, 2)$. Thus

$$\begin{aligned}\text{Int}(A) \cup \text{Int}(B) &= (0, 1) \cup (1, 2) \\ &= \text{Int}(A \cup B) - \{1\}.\end{aligned}$$

Solution to Exercise 4.4

Let $q \in \mathbb{Q}$; then by the density property of the reals, for each $r > 0$, the ball $B_d(q, r) \cap \mathbb{Q}^c \neq \emptyset$ and so, by Theorem 4.4, $q \notin \text{Int}(\mathbb{Q})$. Since $\text{Int}(\mathbb{Q}) \subseteq \mathbb{Q}$, we conclude that $\text{Int}(\mathbb{Q}) = \emptyset$.

Solution to Exercise 4.5

A set A is open if, and only if, A^c is closed – which, by Theorem 3.3, holds if, and only if, $A^c = \text{Cl}(A^c)$. But, by Theorem 4.6, $[\text{Cl}(A^c)]^c = \text{Int}(A)$ and so A is open if, and only if,

$$A = (A^c)^c = [\text{Cl}(A^c)]^c = \text{Int}(A).$$

Solution to Exercise 5.1

By Theorem 5.2,

$$\begin{aligned}\text{Bd}(A^c) &= \text{Cl}(A^c) \cap \text{Cl}((A^c)^c) \\ &= \text{Cl}(A^c) \cap \text{Cl}(A) = \text{Bd}(A).\end{aligned}$$

Solution to Exercise 5.2

(a) By Worked Exercise 3.9, $\text{Cl}(\mathbb{Q}) = \text{Cl}(\mathbb{Q}^c) = \mathbb{R}$. It follows from Theorem 4.6 that

$$\text{Int}(\mathbb{Q}) = [\text{Cl}(\mathbb{Q}^c)]^c = \mathbb{R}^c = \emptyset$$

and

$$\text{Int}(\mathbb{Q}^c) = [\text{Cl}(\mathbb{Q})]^c = \mathbb{R}^c = \emptyset.$$

Hence by Theorem 5.5,

$$\text{Bd}(\mathbb{Q}) = (\text{Int}(\mathbb{Q}^c) \cup \text{Int}(\mathbb{Q}))^c = \emptyset^c = \mathbb{R}.$$

(b) For $\mathbb{Q}^c$, similar arguments can be used to obtain $\text{Bd}(\mathbb{Q}^c) = \mathbb{R}$.

Solution to Exercise 5.3

For the discrete metric, every subset of X is both open and closed. So A and A^c are open and, by Exercise 4.5, $\mathrm{Int}(A) = A$ and $\mathrm{Int}(A^c) = A^c$.

By Theorem 5.5,

$$\begin{aligned}\mathrm{Bd}(A) &= (\mathrm{Int}(A^c) \cup \mathrm{Int}(A))^c \\ &= (A^c \cup A)^c = X^c = \emptyset.\end{aligned}$$

Solution to Exercise 5.4

We note that

$$\begin{aligned}x \in (E \cup F) \cap G &\text{ if, and only if, } x \in (E \cup F) \text{ and } x \in G \\ &\text{if, and only if, } (x \in E \text{ or } x \in F) \text{ and } x \in G \\ &\text{if, and only if, } (x \in E \text{ and } x \in G) \text{ or } (x \in F \text{ and } x \in G) \\ &\text{if, and only if, } (x \in E \cap G) \text{ or } (x \in F \cap G) \\ &\text{if, and only if, } x \in (E \cap G) \cup (F \cap G),\end{aligned}$$

as required.

Solution to Exercise 5.5

It follows from the result of Exercise 5.1 that

$$\mathrm{Bd}(A \cap B) = \mathrm{Bd}((A \cap B)^c).$$

Using De Morgan's Second Law, this gives

$$\mathrm{Bd}(A \cap B) = \mathrm{Bd}(A^c \cup B^c).$$

It now follows from Theorem 5.6 and the result of Exercise 5.1 that

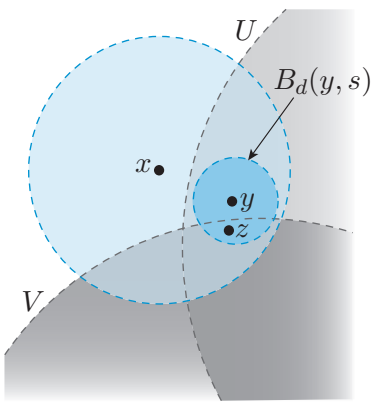
$$\mathrm{Bd}(A \cap B) \subseteq \mathrm{Bd}(A^c) \cup \mathrm{Bd}(B^c) = \mathrm{Bd}(A) \cup \mathrm{Bd}(B).$$

Solution to Exercise 6.1

Since $B \subseteq A$, we have $\mathrm{Cl}(B) \subseteq \mathrm{Cl}(A)$ and so

$$\mathrm{Int}(\mathrm{Cl}(B)) \subseteq \mathrm{Int}(\mathrm{Cl}(A)) = \emptyset.$$

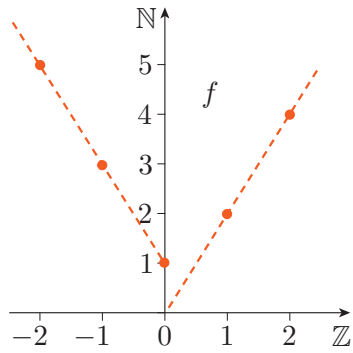
Solution to Exercise 6.2



Since U and V are open, so is $U \cap V$. So we need only show that $U \cap V$ is dense in X – that is, $\mathrm{Cl}(U \cap V) = X$. We do this by using the characterisation of closed sets in terms of closure points, Theorem 3.8, and show that for each $x \in X$ and $r > 0$, $B_d(x, r)$ meets $U \cap V$.

Let $x \in X$ and let $r > 0$ be given. Since U is dense in X, $B_d(x, r) \cap U \neq \emptyset$ and we can choose $y \in B_d(x, r) \cap U$. Moreover, since $B_d(x, r) \cap U$ is open, we can find $s > 0$ so that $B_d(y, s) \subseteq B_d(x, r) \cap U$. But V is dense in X and so $\mathrm{Cl}(V) = X$. In particular, $B_d(y, s) \cap V \neq \emptyset$ and we can choose $z \in B_d(y, s) \cap V$ (as shown). But then $z \in U \cap V \cap B_d(x, r)$ and so $B_d(x, r)$ meets $U \cap V$. We deduce that $x \in \mathrm{Cl}(U \cap V)$ and so $U \cap V$ is dense in X as required.

Solution to Exercise 6.3



The listing of the elements of $\mathbb{Z}$ as

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

leads to the one–one map from $\mathbb{Z}$ to $\mathbb{N}$ defined by

$$f(n) = \begin{cases} 1, & \text{for } n = 0, \\ 2n, & \text{for } n > 0, \\ -2n + 1, & \text{for } n < 0, \end{cases}$$

as shown in the figure. (There are other possible maps.) So $\mathbb{Z}$ is countable.

Solution to Exercise 6.4

Suppose that (a, b) is countable, so that there is a one–one map $f\colon (a, b) \to \mathbb{N}$. Consider the map $g\colon (0, 1) \to (a, b)$ given by

$$g(x) = a + (b - a)x.$$

This map is one–one since if $g(x) = g(y)$, so that $a + (b - a)x = a + (b - a)y$, then $(b - a)x = (b - a)y$; hence, since $a \neq b$, $x = y$. Therefore the map $f \circ g$ is a one–one map from $(0, 1)$ to $\mathbb{N}$. But this is impossible, since $(0, 1)$ is uncountable (by Theorem 6.7). Hence (a, b) must be uncountable.

Solution to Exercise 7.1

Observe that $x \in f^{-1}(Y - A)$ if, and only if, $f(x) \in Y - A$; and $f(x) \in Y - A$ if, and only if, $f(x) \in Y$ and $f(x) \notin A$, which holds if, and only if, $x \in f^{-1}(Y)$ and $x \notin f^{-1}(A)$ – in other words if, and only if, $x \in X - f^{-1}(A)$.

Hence $x \in f^{-1}(Y - A)$ if, and only if, $x \in X - f^{-1}(A)$ and so $f^{-1}(Y - A) = X - f^{-1}(A)$ as required.